

MORIN SINGULARITIES OF COFRAMES AND FRAMES

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ABSTRACT. Inspired by the properties of an n -frame of gradients $(\nabla f_1, \dots, \nabla f_n)$ of a Morin map $f : M \rightarrow \mathbb{R}^n$, with $\dim M \geq n$, we introduce the notion of Morin singularities in the context of singular n -coframes and singular n -frames. We also study the singularities of generic 1-forms associated to a Morin n -coframe, in order to generalize a result of T. Fukuda [4, Theorem 1], which establishes a modulo 2 congruence between the Euler characteristic of a compact manifold M and the Euler characteristics of the singular sets of a Morin map defined on M , to the case of Morin n -coframes and Morin n -frames.

1. INTRODUCTION

Morin maps are maps that only admit Morin singularities. It is well known that these singularities are stable, and conversely, that stable map-germs which have corank 1 are Morin singularities. Therefore, Morin singularities are fundamental and frequently arise as singularities of maps from one manifold to another, as observed by K. Saji in [12]. Morin singularities have been studied by many authors in different contexts as [7, 1, 4, 10, 11], and more recently [6, 15, 18, 5, 2, 12, 14, 13, 9]. In particular, papers of J.M. Éliašberg [3], J.R. Quine [8], T. Fukuda [4], O. Saeki [10] and N. Dutertre and T. Fukui [2] investigate relations between the topology of a manifold and the topology of the critical locus of maps with Morin singularities.

Let $f : M^m \rightarrow \mathbb{R}^n$ be a smooth Morin map defined on an m -dimensional Riemannian manifold M , with $m \geq n$. The singular points of $f = (f_1, \dots, f_n)$ are the points $x \in M$, such that the rank of the derivative $df(x)$ is equal to $n - 1$. Then, taking the gradient of each coordinate function f_1, \dots, f_n , we obtain a singular n -frame $(\nabla f_1(x), \dots, \nabla f_n(x))$ defined on M whose singular locus Σ is given by

$$\Sigma = \{x \in M \mid \text{rank}(\nabla f_1(x), \dots, \nabla f_n(x)) = n - 1\}.$$

It is well known that the singular sets of f , $A_k(f)$ and $\overline{A_k(f)}$ ($k = 1, \dots, n - 1$), are submanifolds of M of dimension $n - k$, such that $\overline{A_k(f)} = \cup_{i \geq k} A_i(f)$ and

$$\text{rank } df|_{\overline{A_k(f)}}(x) = \begin{cases} n - k, & \text{if } x \in A_k(f); \\ n - k - 1, & \text{if } x \in \overline{A_{k+1}(f)}; \end{cases}$$

(see [4], [7], [10] for Morin singularities). This means that the intersection of the vector space spanned by $\nabla f_1(x), \dots, \nabla f_n(x)$ with the normal space to $\overline{A_k(f)}$ at x is a subspace of dimension:

$$\dim(\langle \nabla f_1(x), \dots, \nabla f_n(x) \rangle \cap N_x \overline{A_k(f)}) = \begin{cases} k - 1, & \text{if } x \in A_k(f); \\ k, & \text{if } x \in \overline{A_{k+1}(f)}. \end{cases}$$

In particular, if $x \in \overline{A_k(f)}$ then $\langle \nabla f_1(x), \dots, \nabla f_n(x) \rangle \cap N_x \overline{A_k(f)}$.

Furthermore, if $x \in \overline{A_{k+1}(f)}$ and $\{z_1(x), \dots, z_{n-k-1}(x)\}$ is a basis of a vector space supplementary to $\langle \nabla f_1(x), \dots, \nabla f_n(x) \rangle \cap N_x \overline{A_k(f)}$ in $\langle \nabla f_1(x), \dots, \nabla f_n(x) \rangle$

then

$$\dim(\langle z_1(x), \dots, z_{n-k-1}(x) \rangle \cap N_x \overline{A_{k+1}(f)}) = \begin{cases} 0, & \text{if } x \in A_{k+1}(f); \\ 1, & \text{if } x \in \overline{A_{k+2}(f)}. \end{cases}$$

Based on properties of an n -frame of gradients $(\nabla f_1, \dots, \nabla f_n)$ of a Morin map f , in this paper we introduce the notion of Morin singular points of type A_k in the context of singular n -frames that are not necessarily gradients (Definition 2.7) and n -coframes that are not necessarily differentials (Definition 2.6). To do this, in Section 2 we consider an n -coframe $\omega = (\omega_1, \dots, \omega_n)$ with corank 1 (Definition 2.1) defined on a smooth m -dimensional manifold M , with $m \geq n$, and we proceed by induction on k , for $k = 1, \dots, n$, in order to define Morin singular sets $\Sigma^k(\omega)$ and $A_k(\omega)$ (Definitions 2.2, 2.4 and 2.5). We will say that ω is a Morin n -coframe (Definition 2.6) if it admits only Morin singular points, that is, if each singular point $x \in M$ of ω belongs to $A_k(\omega)$, for some $k = 1, \dots, n$ (see Remark 2.2). In particular, we show that the Morin singular sets $A_k(\omega)$ and $\Sigma^k(\omega) = \overline{A_k(\omega)}$ ($k = 1, \dots, n$) are smooth submanifolds of M of dimension $n - k$ (Lemmas 2.2 and 2.5), such that $\overline{A_k(\omega)} = \cup_{i \geq k} A_i(f)$ (Remark 2.1) and in Lemmas 2.7 and 4.5 we exhibit equations that define locally the singular sets $\Sigma^k(\omega)$.

The definition of Morin singularities for n -coframes can be analogously adapted to n -frames as follows. When considering a smooth manifold M , differential 1-forms are naturally dual to vector fields, more specifically, if we fix a Riemannian metric on M then there exists an isomorphism between the tangent and cotangent bundles of M , so that vector fields and 1-forms can be identified. To illustrate this notion, we give some examples of Morin n -frames in the end of Section 2.

Let $L \in \mathbb{R}P^{n-1}$ be a straight line in \mathbb{R}^n and let $\pi_L : \mathbb{R}^n \rightarrow L$ be the orthogonal projection to L . In [4], T. Fukuda applied Morse theory and well known properties of singular sets $A_k(f)$ of a Morin map $f : M \rightarrow \mathbb{R}^n$ to study the critical points of mappings $\pi_L \circ f : M \rightarrow L$ and their restrictions to the singular sets $\pi_L \circ f|_{A_k(f)}$ and $\pi_L \circ f|_{\overline{A_k(f)}}$. Similarly, in Sections 3 and 4 of this paper, we investigate the zeros of a generic 1-form

$$\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$$

associated to a Morin n -coframe $\omega = (\omega_1, \dots, \omega_n)$ and we verify that ξ , $\xi|_{A_k(\omega)}$ and $\xi|_{\overline{A_k(\omega)}}$ have properties that are analogous to the properties of the generic orthogonal projections $\pi_L \circ f(x)$ associated to a Morin map $f = (f_1, \dots, f_n)$ and of their restrictions. More precisely, let $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ and let $\omega = (\omega_1, \dots, \omega_n)$ be a Morin n -coframe defined on a manifold M , in Section 3 we prove that if $p \in M$ is a zero of $\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$ then $p \in \Sigma^1(\omega)$ and p is a zero of $\xi|_{\Sigma^1(\omega)}$ (Lemma 3.1). In Lemma 3.2, we show that, for $k = 0, \dots, n-2$, if $p \in A_{k+1}(\omega)$ then p is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if p is a zero of $\xi|_{\Sigma^k(\omega)}$. And, in Lemma 3.3 we verify that if $p \in A_n(\omega)$ then p is a zero of the restriction $\xi|_{\Sigma^{n-1}(\omega)}$. Let $Z(\xi|_{\Sigma^k(\omega)})$ be the zero set of the restriction of the 1-form ξ to $\Sigma^k(\omega)$, we also prove in Lemmas 3.5 and 3.6 that for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega) = \emptyset$, for $k = 0, \dots, n-2$.

In Section 4, in Lemmas 4.6, 4.7, 4.8 and 4.12, we prove that generically the 1-form $\xi(x)$ and its restrictions $\xi|_{\Sigma^k(\omega)}$ and $\xi|_{A_k(\omega)}$ admit only non-degenerate zeros. We also show that, for $k = 0, \dots, n-2$, if $p \in A_{k+1}(\omega)$ is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, p is a non-degenerate zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if p is a non-degenerate zero of $\xi|_{\Sigma^k(\omega)}$ (Lemmas 4.9 and 4.10). Finally, in Lemma 4.11 we verify that, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, if $p \in A_n(\omega)$ then p is a non-degenerate zero of $\xi|_{\Sigma^{n-1}(\omega)}$.

As a consequence of these results, we obtain a generalization of Fukuda's Theorem [4, Theorem 1] for the case of Morin n -coframes (Theorem 4.1). More precisely, we prove in Theorem 4.1 that if $\omega = (\omega_1, \dots, \omega_n)$ is a Morin n -coframe defined on an m -dimensional compact manifold M then

$$\chi(M) \equiv \sum_{k=1}^n \chi(\overline{A_k(\omega)}) \pmod{2},$$

where $\chi(M)$ denotes the Euler characteristic of M . We end the paper with this generalized theorem, whose proof uses the classical Poincaré-Hopf Theorem for 1-forms.

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2. THE MORIN n -COFRAMES

Let M be a smooth manifold of dimension m and $\omega = (\omega_1, \dots, \omega_n)$ be a (singular) n -coframe, that is, a set of n smooth 1-forms defined on M , with $m \geq n$:

$$\begin{aligned} \omega : M &\rightarrow T^*M^n \\ x &\mapsto (x, \omega_1(x), \dots, \omega_n(x)) \end{aligned}$$

where $T^*M^n = \{(x, \varphi_1, \dots, \varphi_n) \mid x \in M; \varphi_i \in T_x^*M, i = 1, \dots, n\}$ is the “ n -cotangent bundle” of M . Note that T^*M^n is a smooth manifold of dimension $m(n+1)$, because it is locally diffeomorphic to $U \times M_{m,n}(\mathbb{R})$, where $U \subset \mathbb{R}^m$ is an open set and $M_{m,n}(\mathbb{R})$ denotes the space of matrices of dimension $m \times n$ with real coefficients.

Lemma 2.1. *Let $T^*M^{n,n-1} \subset T^*M^n$ be the subset defined by*

$$T^*M^{n,n-1} = \{(x, \varphi_1, \dots, \varphi_n) \in T^*M^n \mid \text{rank}(\varphi_1, \dots, \varphi_n) = n-1\}.$$

*Then $T^*M^{n,n-1}$ is a submanifold of T^*M^n of dimension $n(m+1) - 1$.*

Proof. Let $M_{m,n}^{n-1}(\mathbb{R})$ be the submanifold of $M_{m,n}(\mathbb{R})$ of codimension $m-n+1$ given by the matrices of rank $n-1$ in $M_{m,n}(\mathbb{R})$, then $T^*M^{n,n-1}$ is locally diffeomorphic to $U \times M_{m,n}^{n-1}(\mathbb{R})$, for some open subset $U \subset \mathbb{R}^m$. Hence, $T^*M^{n,n-1}$ is a submanifold of T^*M^n and $\dim(T^*M^{n,n-1}) = n(m+1) - 1$. \square

Definition 2.1. *We say that $\omega = (\omega_1, \dots, \omega_n)$ has corank 1 if the following properties hold:*

- (a) $\omega \pitchfork T^*M^{n,n-1}$ in T^*M^n ;
- (b) $\omega^{-1}(T^*M^{n,\leq n-2}) = \emptyset$;

*where $T^*M^{n,\leq n-2} = \{(x, \varphi_1, \dots, \varphi_n) \in T^*M^n \mid \text{rank}(\varphi_1, \dots, \varphi_n) \leq n-2\}$.*

Note that by Definition 2.1, if an n -coframe $\omega = (\omega_1, \dots, \omega_n)$ has corank 1 then, for each $x \in M$, $\text{rank}(\omega_1(x), \dots, \omega_n(x))$ is either equal to n or equal to $n-1$.

Definition 2.2. *Let $\omega = (\omega_1, \dots, \omega_n)$ be an n -coframe with corank 1. The singular set of ω , $\Sigma^1(\omega)$, is the set of points $x \in M$ at which the rank is not maximal, that is*

$$\Sigma^1(\omega) = \{x \in M \mid \text{rank}(\omega_1(x), \dots, \omega_n(x)) = n-1\}.$$

Lemma 2.2. *If ω is an n -coframe with corank 1 then $\Sigma^1(\omega)$ is either the empty set or a submanifold of M of dimension $n-1$.*

Proof. Note that $\Sigma^1(\omega) = \omega^{-1}(T^*M^{n,n-1})$ and that $\omega \pitchfork T^*M^{n,n-1}$. Thus, if $\Sigma^1(\omega) \neq \emptyset$ then $\Sigma^1(\omega)$ is a submanifold of M of codimension $m - n + 1$, that is, $\dim(\Sigma^1(\omega)) = n - 1$. \square

Let $\omega = (\omega_1, \dots, \omega_n) : M \rightarrow T^*M^n$ be an n -coframe with corank 1 defined on an m -dimensional smooth manifold M . Next, we will define the subsets $A_k(\omega)$ and $\Sigma^{k+1}(\omega)$ of M , for $k = 1, \dots, n$. To do this we will proceed by induction on k starting from the definition of the singular set $\Sigma^1(\omega)$.

Notation. Let us denote by $\Sigma^0(\omega)$ the manifold M and by $N_x^*\Sigma^0(\omega) = \{0\}$ the set that contains only the null 1-form of T_x^*M . Moreover, if $S \subset M$ is a smooth submanifold of M , let us denote by N_x^*S the set $N_x^*S = \{\psi \in T_x^*M \mid \psi(T_x S) = 0\}$.

We know that $\Sigma^1(\omega) = \{x \in \Sigma^0(\omega) \mid \text{rank}(\omega_1(x), \dots, \omega_n(x)) = n - 1\}$ and that $\dim(\Sigma^1(\omega)) = n - 1$. In particular,

$$p \in \Sigma^1(\omega) \Rightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^*\Sigma^0(\omega)) = 0,$$

where $\langle \omega_1(p), \dots, \omega_n(p) \rangle$ is the vector subspace of T_p^*M spanned by the 1-forms $\omega_1(p), \dots, \omega_n(p)$.

Let us suppose that $\Sigma^i(\omega)$ is defined for $i = 1, \dots, k-1$ so that $\Sigma^i(\omega)$ is a smooth submanifold of M of dimension $n - i$, $\Sigma^i(\omega) \subset \Sigma^{i-1}(\omega)$ and, for $i = 2, \dots, k-1$,

$$p \in \Sigma^i(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^*\Sigma^{i-1}(\omega)) = i - 1,$$

where $\Sigma^i(\omega)$ is locally given by

$$\mathcal{U} \cap \Sigma^i(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+i}(x) = 0\}$$

and

$$\mathcal{U} \cap \Sigma^{i-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+i-1}(x) = 0\},$$

for some open neighborhood $\mathcal{U} \subset M$ and smooth functions $F_i : \mathcal{U} \rightarrow \mathbb{R}$ whose derivatives $dF_i(x) \in T_x^*M$ are linearly independent for each $x \in \Sigma^i(\omega) \cap \mathcal{U}$. Also, $N_x^*\Sigma^{i-1}(\omega)$ is the vector subspace of T_x^*M spanned by these derivatives, that is, $N_x^*\Sigma^{i-1}(\omega) = \langle dF_1(x), \dots, dF_{m-n+i-1}(x) \rangle$.

We set $r = n - k + 1$ and $(x, \varphi) = (x, \varphi_1, \dots, \varphi_r)$. In order to define $\Sigma^k(\omega)$ we first consider:

$$T_{\Sigma^{k-1}}^*M^r = \{(x, \varphi) \mid x \in \Sigma^{k-1}(\omega); \varphi_1, \dots, \varphi_r \in T_x^*M\}$$

and

$$N_{\Sigma^{k-1}}^*M^r = \{(x, \varphi) \in T_{\Sigma^{k-1}}^*M^r \mid \text{rank}(\varphi_1, \dots, \varphi_r) = r, \\ \dim(\langle \varphi_1, \dots, \varphi_r \rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 1\}.$$

Lemma 2.3. $T_{\Sigma^{k-1}}^*M^r$ is a smooth manifold of dimension $mr + r$.

Proof. By the induction hypothesis, $\Sigma^{k-1}(\omega)$ is a smooth submanifold of M of dimension r . Then, there exists an open subset $V \subset \mathbb{R}^r$ so that $T_{\Sigma^{k-1}}^*M^r$ is locally diffeomorphic to $V \times M_{m,r}(\mathbb{R})$. Thus, $T_{\Sigma^{k-1}}^*M^r$ is a smooth manifold and $\dim(T_{\Sigma^{k-1}}^*M^r) = mr + r$. \square

Lemma 2.4. $N_{\Sigma^{k-1}}^*M^r$ is a hypersurface of $T_{\Sigma^{k-1}}^*M^r$, that is, a submanifold of dimension $mr + r - 1$.

Proof. By the induction hypothesis, for each $p \in \Sigma^{k-1}(\omega)$, there exist an open neighborhood $\mathcal{U} \subset M$ of p and functions $F_1, \dots, F_{m-r} : \mathcal{U} \rightarrow \mathbb{R}$ such that

$$\mathcal{U} \cap \Sigma^{k-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = 0\}$$

with $\text{rank}(dF_1(x), \dots, dF_{m-r}(x)) = m - r$, for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}$.

If $(p, \tilde{\varphi}) \in N_{\Sigma^{k-1}}^* M^r$ then $\text{rank}(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) = r$ and

$$\text{rank}(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r, dF_1(p), \dots, dF_{m-r}(p)) = m - 1$$

since $N_p^* \Sigma^{k-1}(\omega) = \langle dF_1(p), \dots, dF_{m-r}(p) \rangle$. Thus,

$$\det(dF_1(p), \dots, dF_{m-r}(p), \tilde{\varphi}_1, \dots, \tilde{\varphi}_r) = 0$$

and fixing the notation $\tilde{\varphi}_i = (\tilde{\varphi}_i^1, \dots, \tilde{\varphi}_i^m)$ for $i = 1, \dots, r$, we can suppose without loss of generality that

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_{m-r}}{\partial x_1}(p) & \tilde{\varphi}_1^1 & \cdots & \tilde{\varphi}_{r-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x_{m-1}}(p) & \cdots & \frac{\partial F_{m-r}}{\partial x_{m-1}}(p) & \tilde{\varphi}_1^{m-1} & \cdots & \tilde{\varphi}_{r-1}^{m-1} \end{vmatrix} \neq 0$$

and consequently, that

$$(1) \quad \begin{vmatrix} \frac{\partial F_1}{\partial x_1}(x) & \cdots & \frac{\partial F_{m-r}}{\partial x_1}(x) & \varphi_1^1 & \cdots & \varphi_{r-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x_{m-1}}(x) & \cdots & \frac{\partial F_{m-r}}{\partial x_{m-1}}(x) & \varphi_1^{m-1} & \cdots & \varphi_{r-1}^{m-1} \end{vmatrix} \neq 0$$

for all $(x, \varphi) \in (\Sigma^{k-1}(\omega) \cap \mathcal{U}) \times \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^{mr}$ is an open subset. Thus, $N_{\Sigma^{k-1}}^* M^r$ can be locally defined by

$$(2) \quad N_{\Sigma^{k-1}}^* M^r = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \dots = F_{m-r} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$.

Let $B(x, \varphi)$ be the square matrix of order m whose columns are given by the coefficients of the 1-forms $dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r$:

$$B(x, \varphi) = \begin{pmatrix} dF_1(x) & \cdots & dF_{m-r}(x) & \varphi_1 & \cdots & \varphi_r \end{pmatrix}.$$

We have,

$$\Delta(x, \varphi) = \sum_{i=1}^m \varphi_r^i \text{cof}(\varphi_r^i, B),$$

where $\text{cof}(\varphi_r^i, B)$ denotes the cofactor of φ_r^i in the matrix $B(x, \varphi)$ so that

$$\frac{\partial \Delta}{\partial \varphi_r^m}(x, \varphi) = \sum_{i=1}^m \text{cof}(\varphi_r^i, B) \frac{\partial \varphi_r^i}{\partial \varphi_r^m} + \varphi_r^i \frac{\partial \text{cof}(\varphi_r^i, B)}{\partial \varphi_r^m}$$

and since $\text{cof}(\varphi_r^i, B)$ does not depend on the variable φ_r^m ,

$$\frac{\partial \text{cof}(\varphi_r^i, B)}{\partial \varphi_r^m} = 0, \text{ for } i = 1, \dots, m.$$

Then,

$$\frac{\partial \Delta}{\partial \varphi_r^m}(x, \varphi) = \text{cof}(\varphi_r^m, B) \stackrel{(1)}{\neq} 0,$$

and the derivative of $\Delta(x, \varphi)$ with respect to φ does not vanish, that is, $d_\varphi \Delta(x, \varphi) \neq 0$ and the matrix

$$\begin{bmatrix} dF_1(x) \\ \vdots \\ dF_{m-r}(x) \\ d\Delta(x, \varphi) \end{bmatrix} = \begin{bmatrix} d_x F_1(x) & \vdots & \\ \vdots & \vdots & O_{(m-r) \times (r)} \\ d_x F_{m-r}(x) & \vdots & \\ \dots & \dots & \dots \\ d_x \Delta(x, \varphi) & \vdots & d_\varphi \Delta(x, \varphi) \end{bmatrix}$$

has rank $m - r + 1$, where $O_{(m-r) \times (r)}$ denotes a null matrix. Hence,

$$\text{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta(x, \varphi)) = m - r + 1,$$

for each $(x, \varphi) \in N_{\Sigma^{k-1}}^* M^r \cap (\mathcal{U} \times \mathcal{V})$ and, therefore, $N_{\Sigma^{k-1}}^* M^r$ is a smooth submanifold of $T_{\Sigma^{k-1}}^* M^r$ of dimension $m + mr - (m - r + 1) = mr + r - 1$. \square

By the induction hypothesis, we have that for each $p \in \Sigma^{k-1}(\omega)$,

$$\dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^{k-2}(\omega)) = k - 2$$

and there exist an open neighborhood $\mathcal{U} \subset M$ of p and functions $F_1, \dots, F_{m-r} : \mathcal{U} \rightarrow \mathbb{R}$ such that $\mathcal{U} \cap \Sigma^{k-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = 0\}$ with $\text{rank}(dF_1(x), \dots, dF_{m-r}(x)) = m - r$, for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}$. Then, we can choose $\{\Omega_1(x), \dots, \Omega_r(x)\}$ a smooth r -coframe defined on \mathcal{U} which restriction to $\mathcal{U} \cap \Sigma^{k-1}(\omega)$ is a smooth basis of a vector subspace supplementary to

$$(3) \quad \langle \omega_1(x), \dots, \omega_n(x) \rangle \cap N_x^* \Sigma^{k-2}(\omega)$$

in $\langle \omega_1(x), \dots, \omega_n(x) \rangle$. Let $\Omega^{k-1} : \Sigma^{k-1}(\omega) \cap \mathcal{U} \rightarrow T_{\Sigma^{k-1}}^* M^r$ be the map given by $\Omega^{k-1}(x) = (x, \Omega_1(x), \dots, \Omega_r(x))$, we define:

Definition 2.3. We say that the n -coframe $\omega = (\omega_1, \dots, \omega_n)$ satisfies the “intersection properties I_k ”, if for each $p \in \Sigma^{k-1}(\omega)$ there exist an open neighborhood $\mathcal{U} \subset M$ of p and a map $\Omega^{k-1} : \Sigma^{k-1}(\omega) \cap \mathcal{U} \rightarrow T_{\Sigma^{k-1}}^* M^r$ as defined above, such that on \mathcal{U} the following properties hold:

- (a) $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$;
- (b) $(\Omega^{k-1})^{-1}(N_{\Sigma^{k-1}}^* M^{r, \geq 2}) = \emptyset$;

where $N_{\Sigma^{k-1}}^* M^{r, \geq 2} = \{(x, \varphi) \in T_{\Sigma^{k-1}}^* M^r \mid \text{rank}(\varphi_1, \dots, \varphi_r) = r, \dim(\langle \varphi_1, \dots, \varphi_r \rangle \cap N_x^* \Sigma^{k-1}(\omega)) \geq 2\}$.

Note that, if the n -coframe ω satisfies the properties I_k (a) and (b) then, for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}$, $\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega))$ is either equal to 0 or equal to 1.

Definition 2.4. Let ω be an n -coframe with corank 1 that satisfies the intersection properties I_k (a) and (b). We say that a point $x \in \Sigma^{k-1}(\omega)$ belongs to $A_{k-1}(\omega)$ if

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 0;$$

and we say that x belongs to $\Sigma^k(\omega)$ if $x \in \Sigma^{k-1}(\omega) \setminus A_{k-1}(\omega)$, that is, if

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 1.$$

Therefore,

$$\begin{aligned} A_{k-1}(\omega) &= \{x \in \Sigma^{k-1}(\omega) \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 0\}; \\ \Sigma^k(\omega) &= \{x \in \Sigma^{k-1}(\omega) \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 1\}. \end{aligned}$$

Definition 2.5. Let ω be an n -coframe with corank 1 that satisfies the intersection properties I_k (a) and (b). We say that a point $x \in M$ is a Morin singular point of type A_k of the n -coframe ω if $x \in A_k(\omega)$.

Lemma 2.5. By Definition 2.3, $\Sigma^k(\omega)$ is either the empty set or a smooth submanifold of M of dimension $n - k$.

Proof. Note that, locally, $\Sigma^k(\omega) = (\Omega^{k-1})^{-1}(N_{\Sigma^{k-1}}^* M^r)$ and $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$, thus, if $\Sigma^k(\omega) \neq \emptyset$ then $\Sigma^k(\omega)$ is a smooth submanifold of $\Sigma^{k-1}(\omega)$ of codimension 1, that is, $\dim(\Sigma^k(\omega)) = n - k$. \square

Lemma 2.6. If $p \in \Sigma^k(\omega)$ then $\dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^{k-1}(\omega)) = k - 1$.

Proof. For clearer notations, let us write $\langle \bar{\omega}(x) \rangle = \langle \omega_1(x), \dots, \omega_n(x) \rangle$ and $\langle \bar{\Omega}^{k-1}(x) \rangle = \langle \Omega_1(x), \dots, \Omega_r(x) \rangle$. Let $p \in \Sigma^k(\omega)$, since $\Sigma^k(\omega) \subset \Sigma^{k-1}(\omega) \subset \Sigma^{k-2}(\omega)$, there exist an open neighborhood $\mathcal{U} \subset M$ of p and functions $F_1, \dots, F_{m-n+k} : \mathcal{U} \rightarrow \mathbb{R}$ such that the submanifolds $\Sigma^i(\omega)$, $i = k - 2, k - 1, k$, can be locally defined by

$$\mathcal{U} \cap \Sigma^i(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+i}(x) = 0\},$$

where the derivatives $\{dF_1(x), \dots, dF_{m-n+i}(x)\}$ are 1-forms linearly independent for each $x \in \Sigma^i(\omega) \cap \mathcal{U}$ and $N_x^* \Sigma^i(\omega) = \langle dF_1(x), \dots, dF_{m-n+i}(x) \rangle$.

By the way the r -coframe $\{\Omega_1(x), \dots, \Omega_r(x)\}$ has been chosen, for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}$ we have

$$\langle \bar{\omega}(x) \rangle = (\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-2}(\omega)) \oplus \langle \bar{\Omega}^{k-1}(x) \rangle,$$

and since $N_x^* \Sigma^{k-2}(\omega) \subset N_x^* \Sigma^{k-1}(\omega)$, $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ is equal to

$$(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-2}(\omega)) \oplus (\langle \bar{\Omega}^{k-1}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)).$$

Since $p \in \Sigma^k(\omega) \subset \Sigma^{k-1}(\omega)$, we know that $\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{k-2}(\omega)) = k - 2$ and by the definition of $\Sigma^k(\omega)$, $\dim(\langle \bar{\Omega}^{k-1}(p) \rangle \cap N_p^* \Sigma^{k-1}(\omega)) = 1$. Therefore, $\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{k-1}(\omega)) = (k - 2) + 1 = k - 1$. \square

Next, we will show that Definitions 2.3 and 2.4 do not depend on the choice of the basis $\{\Omega_1(x), \dots, \Omega_r(x)\}$. To do this, first we must find equations that define the manifold $\Sigma^k(\omega)$ locally.

Lemma 2.7. Let $p \in \Sigma^{k-1}(\omega)$. There are an open neighborhood $\mathcal{U} \subset M$ of p and functions $F_i : \mathcal{U} \rightarrow \mathbb{R}$, $i = 1, \dots, m - r$, such that

$$\mathcal{U} \cap \Sigma^{k-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = 0\},$$

and a smooth r -coframe defined on \mathcal{U} $\{\Omega_1(x), \dots, \Omega_r(x)\}$ which is a basis of a vector subspace supplementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-2}(\omega)$ in $\langle \bar{\omega}(x) \rangle$ for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$.

Let

$$\Delta_k(x) = \det(dF_1, \dots, dF_{m-r}, \Omega_1, \dots, \Omega_r)(x).$$

Then ω satisfies the intersection properties I_k on \mathcal{U} if and only if the following properties hold:

(i) $\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 0$ or 1 for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$;

(ii) if $\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = 1$ (or equivalently $\Delta_k(x) = 0$), then $\text{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1$.

In this case, $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = \Delta_k(x) = 0\}$.

Proof. First, let us show that for each $\bar{x} \in \mathcal{U} \cap \Sigma^k(\omega)$,

$$\text{rank}(dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x}))$$

is equal to $m - r + 1$ if and only if $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at \bar{x} .

By Lemma 2.4, $N_{\Sigma^{k-1}}^* M^r$ can be locally defined by

$$N_{\Sigma^{k-1}}^* M^r = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \dots = F_{m-r} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$ and $\mathcal{V} \subset \mathbb{R}^{mr}$. Let

$$G(\Omega^{k-1}) = \{(x, \Omega_1(x), \dots, \Omega_r(x)) \mid x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)\}$$

be the restriction of the graph of $(\Omega_1(x), \dots, \Omega_r(x))$ to $\mathcal{U} \cap \Sigma^{k-1}(\omega)$, $G(\Omega^{k-1})$ can be locally defined by

$$G(\Omega^{k-1}) = \{(x, \varphi) \in T^* M^r \mid F_1(x) = \dots = F_{m-r}(x) = 0; \\ \Omega_i^j(x) - \varphi_i^j = 0, i = 1, \dots, r \text{ and } j = 1, \dots, m\},$$

where $T^* M^r$ denotes the r -cotangent bundle of M , $\Omega_i(x) = (\Omega_i^1(x), \dots, \Omega_i^m(x))$ and $\varphi_i = (\varphi_i^1, \dots, \varphi_i^m)$ for $i = 1, \dots, r$. In particular, the local equations of $G(\Omega^{k-1})$ are clearly independent and $\dim G(\Omega^{k-1}) = r$. Let (x, φ) be local coordinates in $T^* M^r$, with $x = (x_1, \dots, x_m)$ and

$$\varphi = (\varphi_1^1, \dots, \varphi_1^m, \varphi_2^1, \dots, \varphi_2^m, \dots, \varphi_r^1, \dots, \varphi_r^m),$$

let us consider the derivatives of the local equations of $N_{\Sigma^{k-1}}^* M^r$ and $G(\Omega^{k-1})$ with respect to (x, φ) . We will denote the derivative with respect to x by d_x and the derivative with respect to φ by d_φ , then we have

$$(4) \quad d(\Omega_i^j(x) - \varphi_i^j) = (d_x \Omega_i^j(x), -d_\varphi \varphi_i^j),$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$, where $d_\varphi \varphi_i^j = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector whose $m(i-1) + j^{\text{th}}$ entry is equal to 1 and the others are zero. By Lagrange's rules the determinant $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$ can be written as

$$\Delta(x, \varphi) = \sum_I F_I(x) N_I(\varphi)$$

for $I = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}$, where

$$(5) \quad N_I(\varphi) = \begin{vmatrix} \varphi_1^{i_1} & \dots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \dots & \varphi_r^{i_r} \end{vmatrix}$$

is the minor obtained from the matrix

$$\begin{bmatrix} \varphi_1^1 & \dots & \varphi_r^1 \\ \vdots & \ddots & \vdots \\ \varphi_1^m & \dots & \varphi_r^m \end{bmatrix}$$

taking the lines i_1, \dots, i_r , and

$$(6) \quad F_I(x) = \pm \begin{vmatrix} \frac{\partial F_1}{\partial x_{k_1}}(x) & \dots & \frac{\partial F_{m-r}}{\partial x_{k_1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x_{k_{m-r}}}(x) & \dots & \frac{\partial F_{m-r}}{\partial x_{k_{m-r}}}(x) \end{vmatrix}$$

is, up to sign, the minor obtained from the matrix $(dF_1(x) \dots dF_{m-r}(x))$ removing the lines i_1, \dots, i_r , that is, $\{k_1, \dots, k_{m-r}\} = \{1, \dots, m\} \setminus I$. Therefore,

$$d\Delta(x, \varphi) = \left(\sum_I N_I(\varphi) d_x F_I(x), \sum_I F_I(x) d_\varphi N_I(\varphi) \right).$$

Note that $\Omega^{k-1} \cap N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at the point $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ if and only if $G(\Omega^{k-1}) \cap N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at $(x, \Omega^{k-1}(x))$. Let π_1 be the projection of the contangent space of $T^* M^r$ over the contangent space of $T_{\Sigma^{k-1}}^* M^r$:

$$\begin{aligned} \pi_1 : \quad T_{(x, \varphi)}^*(T^* M^r) &\longrightarrow T_{(x, \varphi)}^*(T_{\Sigma^{k-1}}^* M^r) \\ (\psi(x), \varphi_1, \dots, \varphi_r) &\longmapsto (\pi(\psi(x)), \varphi_1, \dots, \varphi_r) \end{aligned}$$

where π denotes the restriction to $T_x \Sigma^{k-1}(\omega)$, that is, $\pi(\psi(x)) = \psi(x)|_{T_x \Sigma^{k-1}(\omega)}$. By Equation (4),

$$\pi_1 \left(d(\Omega_i^j(x) - \varphi_i^j) \right) = \left(\pi(d_x \Omega_i^j(x)), -d_\varphi \varphi_i^j \right),$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$. We also have that

$$\pi_1(d\Delta(x, \varphi)) = \left(\pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right), \sum_I F_I(x) d_\varphi N_I(\varphi) \right).$$

Then, $G(\Omega^{k-1}) \cap N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at $(x, \Omega^{k-1}(x))$ if and only if the matrix

$$(7) \quad \begin{bmatrix} \pi(d_x \Omega_1^1(x)) & \vdots & \\ \vdots & \vdots & \\ \pi(d_x \Omega_1^m(x)) & \vdots & -Id_{mr} \\ \vdots & \vdots & \\ \pi(d_x \Omega_r^m(x)) & \vdots & \\ \dots & \vdots & \dots \\ \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) & \vdots & \sum_I F_I(x) d_\varphi N_I(\varphi) \end{bmatrix}$$

has maximal rank at x . By the expression of $N_I(\varphi)$ in (5), we have

$$(8) \quad d_\varphi N_I(\varphi) = \sum_{i,j} \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j,$$

for $i = 1, \dots, r$, $j \in I$ and $\text{cof}(\varphi_i^j)$ denoting the cofactor of φ_i^j in the matrix

$$\begin{bmatrix} \varphi_1^{i_1} & \dots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \dots & \varphi_r^{i_r} \end{bmatrix}.$$

Let $d = C_{m,r} = \frac{m!}{r!(m-r)!}$, we will denote by I_1, \dots, I_d the subsets of $\{1, \dots, m\}$ containing exactly r elements. By equation (8),

$$\sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{\ell=1}^d F_{I_\ell}(x) \left(\sum_{i=1}^r \sum_{j \in I_\ell} \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right)$$

and,

$$\begin{aligned}
& \sum_{\ell=1}^d F_{I_\ell}(x) \left(\sum_{i=1}^r \sum_{j \in I_\ell} \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \\
&= \sum_{i=1}^r \left[F_{I_1}(x) \left(\sum_{j \in I_1} \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) + \dots + F_{I_d}(x) \left(\sum_{j \in I_d} \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \right] \\
&= \sum_{i=1}^r \left[\left(\sum_{I: 1 \in I} F_I(x) \right) \text{cof}(\varphi_i^1) d_\varphi \varphi_i^1 + \dots + \left(\sum_{I: m \in I} F_I(x) \right) \text{cof}(\varphi_i^m) d_\varphi \varphi_i^m \right] \\
&= \sum_{i=1}^r \left[\sum_{j=1}^m \left(\sum_{I: j \in I} F_I(x) \right) \text{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right].
\end{aligned}$$

Thus, for $i = 1, \dots, r$ and $j = 1, \dots, m$, we can write

$$(9) \quad \sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{i,j} \beta_i^j(x, \varphi) d_\varphi \varphi_i^j,$$

where

$$\beta_i^j(x, \varphi) = \left(\sum_{I: j \in I} F_I(x) \right) \text{cof}(\varphi_i^j).$$

We will denote the rows of the Matrix (7) by $R_i^j = \left(\pi(d_x \Omega_i^j(x)), -d_\varphi \varphi_i^j \right)$, for $i = 1, \dots, r$ and $j = 1, \dots, m$, and we denote the last row of the Matrix (7) by R_Δ . Replacing the row R_Δ by

$$R_\Delta + \sum_{i,j} \beta_i^j(x, \varphi) R_i^j$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$, we obtain a new matrix

$$(10) \quad \begin{bmatrix} \pi(d_x \Omega_1^1(x)) & \vdots & \\ \vdots & \vdots & -Id_{mr} \\ \pi(d_x \Omega_r^m(x)) & \vdots & \\ \dots & \vdots & \dots \\ R'_\Delta & \vdots & R''_\Delta \end{bmatrix}$$

which has rank equal to the rank of the Matrix (7), where

$$R''_\Delta = \sum_I F_I(x) d_\varphi N_I(\varphi) + \sum_{i,j} \beta_i^j(x, \varphi) (-d_\varphi \varphi_i^j) \stackrel{(9)}{=} \vec{0}$$

and

$$\begin{aligned}
R'_\Delta &= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) + \sum_{i,j} \beta_i^j(x, \varphi) \pi \left(d_x \Omega_i^j(x) \right) \\
&= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) + \sum_{i,j} \beta_i^j(x, \varphi) d_x \Omega_i^j(x) \right).
\end{aligned}$$

Note that for each $\bar{x} \in \mathcal{U} \cap \Sigma^k(\omega)$, we have $\Omega_i^j(\bar{x}) = \varphi_i^j$. In this case, Equation (9) implies that

$$\sum_{i,j} \beta_i^j(\bar{x}, \varphi) d_x \Omega_i^j(\bar{x}) = \sum_{i,j} \beta_i^j(\bar{x}, \Omega^{k-1}(\bar{x})) d_x \Omega_i^j(\bar{x}) = \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x})).$$

Thus, at \bar{x}

$$R'_\Delta = \pi \left(\sum_I N_I(\Omega^{k-1}(\bar{x})) d_x F_I(\bar{x}) + \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x})) \right) = \pi(d\Delta_k(\bar{x}))$$

and the Matrix (10) is equal to

$$\begin{bmatrix} \pi(d_x \Omega_1^1(\bar{x})) & \vdots & \\ \vdots & \vdots & -Id_{mr} \\ \pi(d_x \Omega_r^m(\bar{x})) & \vdots & \\ \dots & \vdots & \dots \\ \pi(d\Delta_k(\bar{x})) & \vdots & \vec{0} \end{bmatrix}.$$

Thus, for each $\bar{x} \in \mathcal{U} \cap \Sigma^k(\omega)$, $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at \bar{x} if and only if $\pi(d\Delta_k(\bar{x})) \neq 0$, that is, the restriction of $d\Delta_k(\bar{x})$ to $T_{\bar{x}} \Sigma^{k-1}(\omega)$ is not zero, which means that $d\Delta_k(\bar{x}) \notin \langle dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}) \rangle$, or equivalently $\text{rank}(dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x})) = m - r + 1$.

Now suppose that ω satisfies the intersection properties I_k on \mathcal{U} . By property (b) of Definition 2.3, we have that $\dim \langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ is either equal to 0 or equal to 1 for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. If $\dim \langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 1$, then $\Delta_k(x) = 0$ and $x \in \mathcal{U} \cap \Sigma^k(\omega)$. In this case, the transversality given by property (a) of Definition 2.3 implies that $\text{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1$.

On the other hand, we assume that properties (i) and (ii) hold for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. By property (i), the intersection property (b) of Definition 2.3 holds on \mathcal{U} . If $\dim \langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 0$ then $\Omega^{k-1}(x)$ does not intersect $N_{\Sigma^{k-1}}^* M^r$, thus $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at x . If $\dim \langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 1$ then $x \in \mathcal{U} \cap \Sigma^k(\omega)$ by Definition 2.4 and $\text{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1$ by property (ii). Thus $\Omega^{k-1} \pitchfork N_{\Sigma^{k-1}}^* M^r$ in $T_{\Sigma^{k-1}}^* M^r$ at x and ω satisfies the intersection properties I_k on \mathcal{U} .

Finally, if ω satisfies the intersection properties I_k on \mathcal{U} , it follows by Definition 2.4 that $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = \Delta_k(x) = 0\}$. \square

The following technical lemma will be used in the proof of Lemma 2.9.

Lemma 2.8. *Let $f_i : \mathcal{V} \subset \mathbb{R}^\ell \rightarrow \mathbb{R}, i = 1, \dots, s$ be smooth functions defined on an open neighborhood of \mathbb{R}^ℓ . Let $M \subset \mathbb{R}^\ell$ be a manifold given locally by $M = \{x \in \mathcal{V} \mid f_1(x) = \dots = f_s(x) = 0\}$, with $\text{rank}(df_1(x), \dots, df_s(x)) = s$, for all $x \in M \cap \mathcal{V}$. If $g, h : \mathcal{V} \subset \mathbb{R}^\ell \rightarrow \mathbb{R}$ are smooth functions such that $g(x) = \lambda(x)h(x)$, for all $x \in M \cap \mathcal{V}$ and some smooth function $\lambda : \mathcal{V} \rightarrow \mathbb{R}$, then:*

- (i) *If $\lambda(x) \neq 0$ and $x \in M$ then $g(x) = 0 \Leftrightarrow h(x) = 0$.*
- (ii) *If $\lambda(x) \neq 0, x \in M$ and $h(x) = 0$ then*

$$\langle df_1(x), \dots, df_s(x), dg(x) \rangle = \langle df_1(x), \dots, df_s(x), dh(x) \rangle.$$

Lemma 2.9. *The definitions of $\Sigma^{k+1}(\omega)$ and $A_k(\omega)$ do not depend on the choice of the basis $\{\Omega_1, \dots, \Omega_{n-k}\}$, for every $k \geq 1$.*

Proof. As for the definition of $\Sigma^{k+1}(\omega)$ and $A_k(\omega)$, for $k \geq 1$, we will proceed by induction on k . First, note that the definition of $\Sigma^1(\omega)$ does not depend on the choice of any basis. Then, assume as induction hypothesis that the definition of $\Sigma^i(\omega)$ does not depend on the choice of the basis for every $i \leq k$. We know that, for each $p \in \Sigma^k(\omega)$, there is an open neighborhood $\mathcal{U} \subset M$ of p so that

$$\begin{aligned}\mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} : F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}, \\ \mathcal{U} \cap \Sigma^{k+1}(\omega) &= \{x \in \mathcal{U} : F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_{k+1}(x) = 0\},\end{aligned}$$

with $\text{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_k(x)) = m - n + k$, for $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and $\text{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_{k+1}(x)) = m - n + k + 1$, for $x \in \mathcal{U} \cap \Sigma^{k+1}(\omega)$. Let us recall that

$$\Delta_{k+1}(x) = \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_k, \Omega_1, \dots, \Omega_{n-k})(x),$$

where $\{\Omega_1(x), \dots, \Omega_{n-k}(x)\}$ is a smooth $(n - k)$ -coframe defined on \mathcal{U} which is a basis of a vector subspace supplementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ in $\langle \bar{\omega}(x) \rangle$ for each $x \in \mathcal{U} \cap \Sigma^k(\omega)$.

Let us consider $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x)\}$ a smooth $(n - k)$ -coframe defined on \mathcal{U} such that, for each $x \in \mathcal{U} \cap \Sigma^k(\omega)$, $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x)\}$ is another basis of a vector subspace supplementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ in $\langle \bar{\omega}(x) \rangle$. Then,

$$\langle \bar{\omega}(x) \rangle = (\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) \oplus \langle \tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x) \rangle$$

and $\dim(\langle \tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x) \rangle \cap N_x^* \Sigma^k(\omega))$ is either equal to 0 or equal to 1, for $x \in \mathcal{U} \cap \Sigma^k(\omega)$. Moreover,

$$\left\{ \begin{array}{l} \tilde{\Omega}_1(x) = \sum_{\ell=1}^{n-k} a_{\ell 1}(x) \Omega_{\ell}(x) + \varphi_1(x) \\ \tilde{\Omega}_2(x) = \sum_{\ell=1}^{n-k} a_{\ell 2}(x) \Omega_{\ell}(x) + \varphi_2(x) \\ \vdots \\ \tilde{\Omega}_{n-k}(x) = \sum_{\ell=1}^{n-k} a_{\ell(n-k)}(x) \Omega_{\ell}(x) + \varphi_{n-k}(x) \end{array} \right.$$

where $a_{ij}(x) \in \mathbb{R}$ and $\varphi_j(x) \in \langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$, for $j = 1, \dots, n - k$. We will show that for each $x \in \mathcal{U} \cap \Sigma^k(\omega)$,

$$\det(A(x)) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1(n-k)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-k)1}(x) & a_{(n-k)2}(x) & \cdots & a_{(n-k)(n-k)}(x) \end{vmatrix} \neq 0.$$

Suppose that the statement is false, that is, $\det(A(x)) = 0$. This means that the columns of matrix $A(x)$ are linearly dependent. So we can suppose without loss of generality that the first column of $A(x)$ can be written as a linear combination of the others columns:

$$(a_{11}(x), \dots, a_{(n-k)1}(x)) = \sum_{s=2}^{n-k} \lambda_s (a_{1s}(x), \dots, a_{(n-k)s}(x)),$$

where $\lambda_s \in \mathbb{R}$, for $s = 2, \dots, n - k$. Thus, deleting x in the notation, we have

$$\begin{aligned}
\tilde{\Omega}_1 &= \sum_{\ell=1}^{n-k} a_{\ell 1} \Omega_\ell + \varphi_1 \Rightarrow \tilde{\Omega}_1 = \sum_{\ell=1}^{n-k} \left(\sum_{s=2}^{n-k} \lambda_s a_{\ell s} \right) \Omega_\ell + \varphi_1 \\
&\Rightarrow \tilde{\Omega}_1 = \sum_{s=2}^{n-k} \lambda_s \left(\sum_{\ell=1}^{n-k} a_{\ell s} \Omega_\ell \right) + \varphi_1
\end{aligned}$$

then,

$$\begin{aligned}
\tilde{\Omega}_1 - \sum_{s=2}^{n-k} \lambda_s \tilde{\Omega}_s &= \left[\sum_{s=2}^{n-k} \lambda_s \left(\sum_{\ell=1}^{n-k} a_{\ell s} \Omega_\ell \right) + \varphi_1 \right] - \sum_{s=2}^{n-k} \lambda_s \left(\sum_{\ell=1}^{n-k} a_{\ell s} \Omega_\ell + \varphi_s \right) \\
&= \varphi_1 - \sum_{s=2}^{n-k} \lambda_s \varphi_s.
\end{aligned}$$

This means that

$$\tilde{\Omega}_1 - \sum_{s=2}^{n-k} \lambda_s \tilde{\Omega}_s \in (\langle \bar{\omega} \rangle \cap N_x^* \Sigma^{k-1}(\omega)) \cap \langle \tilde{\Omega}_1, \dots, \tilde{\Omega}_{n-k} \rangle = \{0\},$$

that is, $\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x)$ are linearly dependent. However, this contradicts the initial assumption that $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x)\}$ is a basis of a vector subspace for each x in $\mathcal{U} \cap \Sigma^k(\omega)$. Therefore, $\det(A(x)) \neq 0$.

Let ${}^tA(x)$ be the transpose of matrix $A(x)$. For each $x \in \mathcal{U} \cap \Sigma^k(\omega)$, we have $\det({}^tA(x)) = \det(A(x)) \neq 0$ and, deleting x in the notation,

$$\begin{aligned}
\tilde{\Delta}_{k+1} &= \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_k, \tilde{\Omega}_1, \dots, \tilde{\Omega}_{n-k}) \\
&= \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_k, \sum_{\ell=1}^{n-k} a_{\ell 1} \Omega_\ell, \dots, \sum_{\ell=1}^{n-k} a_{\ell(n-k)} \Omega_\ell) \\
&= \det({}^tA) \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_k, \Omega_1, \dots, \Omega_{n-k}) \\
&= \det({}^tA) \Delta_{k+1}.
\end{aligned}$$

So, by statement (i) of Lemma 2.8, $\tilde{\Delta}_{k+1}(x) = 0 \Leftrightarrow \Delta_{k+1}(x) = 0$ for $x \in \mathcal{U} \cap \Sigma^k(\omega)$. Since $\Delta_{k+1}(x) = 0$ if and only if $\dim(\langle \Omega_1(x), \dots, \Omega_{n-k}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = 1$ and $\tilde{\Delta}_{k+1}(x) = 0$ if and only if $\dim(\langle \tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = 1$, by Definition 2.4 we have that

$$\begin{aligned}
x \in \mathcal{U} \cap \Sigma^{k+1}(\omega) &\Leftrightarrow \dim(\langle \Omega_1(x), \dots, \Omega_{n-k}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = 1 \\
&\Leftrightarrow \Delta_{k+1}(x) = 0 \\
&\Leftrightarrow \tilde{\Delta}_{k+1}(x) = 0 \\
&\Leftrightarrow \dim(\langle \tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = 1
\end{aligned}$$

In particular, if $x \in \mathcal{U} \cap \Sigma^{k+1}(\omega)$ we have $\Delta_{k+1}(x) = 0$ and $\tilde{\Delta}_{k+1}(x) = 0$ so that by statement (ii) of Lemma 2.8,

$$\begin{aligned}
&\langle dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_k(x), d\Delta_{k+1}(x) \rangle \\
&= \langle dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_k(x), d\tilde{\Delta}_{k+1}(x) \rangle,
\end{aligned}$$

which implies that $\text{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_k(x), d\tilde{\Delta}_{k+1}(x))$ is equal to $m - n + k + 1$. Therefore, the intersection properties I_{k+1} and the definition of $\Sigma^{k+1}(\omega)$ do not depend on the choice of the basis $\{\Omega_1(x), \dots, \Omega_{n-k}(x)\}$. Since $A_k(\omega) = \Sigma^k(\omega) \setminus \Sigma^{k+1}(\omega)$, we conclude that $A_k(\omega)$ also does not depend on the choice of the basis. \square

Remark 2.1. It is not difficult to see that $\Sigma^k(\omega)$ is a closed submanifold of M , for $k \geq 1$. Moreover, we can write

$$\Sigma^k(\omega) = A_k(\omega) \cup \Sigma^{k+1}(\omega) = \cup_{i \geq k} A_i(\omega)$$

so that $A_k(\omega) = \Sigma^k(\omega) \setminus \Sigma^{k+1}(\omega)$. That is, the singular sets $A_k(\omega)$ are $(n - k)$ -dimensional submanifolds of M such that $\overline{A_k(\omega)} = \Sigma^k(\omega)$.

Finally, based on the previous considerations, we define:

Definition 2.6. An n -coframe ω is a Morin n -coframe if ω has corank 1 and it satisfies the intersection properties I_k (a) and (b) for $k = 2, \dots, n$.

Remark 2.2. By Definition 2.6, if ω is a Morin n -coframe then ω admits only singular points of type A_k for $k = 1, \dots, n$.

As we mentioned in Section 1, fixed a Riemannian metric on M , we can consider vector fields instead of 1-forms and define the notion of Morin n -frames analogously to the definition of Morin n -coframes:

Definition 2.7. An n -frame $V = (V_1, \dots, V_n) : M \rightarrow TM^n$ is a Morin n -frame if V has corank 1 and it satisfies the intersection properties I_k (a) and (b) for $k = 2, \dots, n$.

Next, we present some examples of Morin n -frames.

Example 2.3. Let $f : M^m \rightarrow \mathbb{R}^n$ be a smooth Morin map defined on an m -dimensional Riemannian manifold M , with $m \geq n$. The n -frame $V(x) = (\nabla f_1(x), \dots, \nabla f_n(x))$ given by the gradient of the coordinate functions of f is, clearly, a Morin n -coframe whose singular points are the same that the singular points of f . That is, $A_k(V) = A_k(f)$, $\forall k = 1, \dots, n$.

Example 2.4. Let $a \in \mathbb{R}$ be a regular value of a C^2 mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. Suppose that $M = f^{-1}(a)$ and consider $V = (V_1, V_2)$ be a 2-frame on M , given by

$$\begin{aligned} V_1(x) &= (-f_{x_2}(x), f_{x_1}(x), 0); \\ V_2(x) &= (-f_{x_3}(x), 0, f_{x_1}(x)). \end{aligned}$$

Since a is a regular value of f , we have that $\nabla f(x) = (f_{x_1}(x), f_{x_2}(x), f_{x_3}(x)) \neq \vec{0}$, $\forall x \in M$. Thus, $\text{rank}(V_1(x), V_2(x))$ is either equal to 2 or equal to 1. The singular points of V are the points $x \in M$ where $\text{rank}(V_1(x), V_2(x)) = 1$, that is,

$$\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$$

and $V = (V_1, V_2)$ has corank 1 if and only if $\text{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ for each $x \in \Sigma^1(V)$. In this case, $\Sigma^1(V)$ is a submanifold of M of dimension 1. Let $x \in \Sigma^1(V)$ be a singular point of V , then the space $\langle V_1(x), V_2(x) \rangle$ is spanned by the vector $e_1 = (1, 0, 0)$ and $x \in A_2(V)$ if and only if

$$\text{rank}(\nabla f(x), \nabla f_{x_1}(x), e_1) < 3,$$

that is, if and only if $\Delta_2 := f_{x_2}f_{x_1x_3} - f_{x_3}f_{x_1x_2}$ vanishes at x . Moreover, V satisfies the intersection properties I_2 if and only if $\text{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) = 3$ for $x \in A_2(V)$. In this case, $A_2(V)$ is a submanifold of M of dimension 0. Therefore, $V = (V_1, V_2)$ is a Morin 2-frame if and only if $\text{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ on the singular set $\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$ and $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) \neq 0$ on $A_2(V) = \{x \in M \mid f_{x_1}(x) = 0, \Delta_2(x) = 0\}$.

Example 2.5. Let us apply Example 2.4 to the 2-frame $V = (V_1, V_2)$ defined on the torus $T := f^{-1}(R^2)$, where R^2 is a regular value of

$$f(x_1, x_2, x_3) = (\sqrt{x_2^2 + x_3^2} - a)^2 + (x_1 + x_2)^2,$$

with $a > R$. Then, one can verify that $\Sigma^1(V) = \{x \in T \mid x_1 + x_2 = 0\}$, that is,

$$\Sigma^1(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sqrt{x_2^2 + x_3^2} - a)^2 = R^2\}$$

and $\text{rank}(\nabla f(x), \nabla f_{x_1}(x))$ is equal to

$$\text{rank} \begin{bmatrix} 0 & \frac{2x_2(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}} & \frac{2x_3(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}} \\ 1 & 1 & 0 \end{bmatrix}$$

which is 2, for all $x \in T \cap \Sigma^1(V)$. Moreover,

$$\Delta_2(x) = \frac{-4x_3(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}},$$

so that $A_2(V) = \{x \in T \mid x_1 + x_2 = 0; x_3 = 0\}$ which is the set given by the points $(-a - R, a + R, 0)$, $(a + R, -a - R, 0)$, $(-a + R, a - R, 0)$ and $(a - R, -a + R, 0)$. It is not difficult to see that $\text{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) = 3, \forall x \in T \cap A_2(V)$. Therefore, the frame $V = (V_1, V_2)$ given by

$$\begin{aligned} V_1(x) &= \left(\frac{-2x_2(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}} - 2(x_1 + x_2), 2(x_1 + x_2), 0 \right); \\ V_2(x) &= \left(\frac{-2x_3(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}}, 0, 2(x_1 + x_2) \right). \end{aligned}$$

is a Morin 2-frame defined on the torus T which admits singular points of type A_1 and A_2 .

Example 2.6. Let $a \in \mathbb{R}$ be a regular value of a C^2 mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. Suppose that $M = f^{-1}(a)$ and consider \overline{W}_1 and \overline{W}_2 be the orthogonal projections of $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ over $T_x M$ given by

$$\begin{aligned} \overline{W}_1 &= e_2 - \left\langle e_2, \frac{\nabla f}{|\nabla f|} \right\rangle \frac{\nabla f}{|\nabla f|}; \\ \overline{W}_2 &= e_3 - \left\langle e_3, \frac{\nabla f}{|\nabla f|} \right\rangle \frac{\nabla f}{|\nabla f|}. \end{aligned}$$

Let $W = (W_1, W_2)$ be the 2-frame defined by $W_1 = \|\nabla f\|^2 \overline{W}_1$ and $W_2 = \|\nabla f\|^2 \overline{W}_2$, that is,

$$\begin{aligned} W_1 &= (-f_{x_1}f_{x_2}, f_{x_1}^2 + f_{x_3}^2, -f_{x_2}f_{x_3}); \\ W_2 &= (-f_{x_1}f_{x_3}, -f_{x_2}f_{x_3}, f_{x_1}^2 + f_{x_2}^2). \end{aligned}$$

Note that in this case, W_1 and W_2 are gradients vector fields, that is, W is a 2-frame gradient. It is not difficult to see that $\text{rank}(W_1(x), W_2(x))$ is either equal to 2 or equal to 1 and the singular set of W is $\Sigma^1(W) = \{x \in M \mid f_{x_1}(x) = 0\}$. Let $x \in \Sigma^1(W)$ be a singular point of W , then the space $\langle W_1(x), W_2(x) \rangle$ is spanned by the vector $(0, f_{x_3}, -f_{x_2})$, so that $A_2(W) = \{x \in M \mid f_{x_1}(x) = 0, f_{x_1x_1}(x) = 0\}$. Therefore, $W = (W_1, W_2)$ is a Morin 2-frame if and only if $\text{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ on the singular set $\Sigma^1(W)$ and $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla f_{x_1x_1}(x)) \neq 0$ on $A_2(W)$.

Example 2.7. Let us apply Example 2.6 to the 2-frame $W = (W_1, W_2)$ defined on the torus $T := f^{-1}(R^2)$ of Example 2.5. In this situation, one can verify that $\Sigma^1(W)$ is the same singular set as $\Sigma^1(V)$ in the Example 2.5. Moreover, $\text{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2, \forall x \in \Sigma^1(W)$. However, since $f_{x_1x_1}(x) = 2, \forall x \in \Sigma^1(W)$, we have that W does not admits singular points of type A_2 . That is, W is Morin 2-frame on T which admits only Morin singularities of type A_1 .

Example 2.8. Let us consider the 2-frames $V = (V_1, V_2)$ and $W = (W_1, W_2)$ from Examples 2.4 and 2.6 defined on the unit sphere $M := f^{-1}(1)$, where $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. We know that the singular sets of V and W are the same, that is, $\Sigma^1(V) = \Sigma^1(W) = \{x \in M \mid x_1 = 0\}$ and $\text{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ for all singular

point x . However, $\Delta_2(x) = 0, \forall x \in \Sigma^1(V)$, so that $\nabla\Delta_2 \equiv \vec{0}$. On the other hand, $f_{x_1x_1}(x) \neq 0, \forall x \in \Sigma^1(W)$, so that $A_2(W) = \emptyset$. Therefore, V is not a Morin 2-frame and W is a Morin 2-frame that admits only Morin singularities of type A_1 .

Example 2.9. In the Example 2.8, if we consider $f(x_1, x_2, x_3) = x_1^2 - x_1x_2 + x_3^2$ then one can verify that V and W are both Morin 2-frames that admits only Morin singularities of type A_1 . Let us consider the case where V of Example 2.4 is defined on $M := f^{-1}(-1)$ and $f(x_1, x_2, x_3) = x_1^2 - x_1x_2 + x_3^2$. It is easy to see that -1 is a regular value of f and $\Sigma^1(V) = \{x \in M \mid 2x_1 - x_2 = 0\}$. That is,

$$\Sigma^1(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0\}$$

and $\text{rank}(\nabla f(x), \nabla f_{x_1}(x))$ is equal to

$$\text{rank} \begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \end{bmatrix}$$

which is 2, for all $x \in M \cap \Sigma^1(V)$. Moreover, $\Delta_2(x) = 2x_3$ and

$$A_2(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0; x_3 = 0\}$$

which is the set given by the points $(1, 2, 0)$ and $(-1, -2, 0)$. We also have that $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla\Delta_2(x))$ is equal to

$$\det \begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 4x_1$$

which is equal to ± 4 for each $x \in A_2(V)$. That is, $\text{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla\Delta_2(x)) = 3, \forall x \in M \cap A_2(V)$. Therefore, the 2-frame $V = (V_1, V_2)$ given by

$$\begin{aligned} V_1(x) &= (x_1, 2x_1 - x_2, 0); \\ V_2(x) &= (-2x_3, 0, 2x_1 - x_2). \end{aligned}$$

is a Morin 2-frame defined on M which admits singular points of type A_1 and A_2 .

3. ZEROS OF A GENERIC 1-FORM $\xi(x)$ ASSOCIATED TO A MORIN n -COFRAME

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ and let $\omega = (\omega_1, \dots, \omega_n)$ be a Morin n -coframe defined on an m -dimensional manifold M . In this section, we will consider the 1-form $\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$ defined on M and we will proof some properties of the zeros of ξ and its restrictions to the singular sets of ω .

Lemma 3.1. *If p is a zero of the 1-form ξ then $p \in \Sigma^1(\omega)$ and p is a zero of $\xi|_{\Sigma^1(\omega)}$.*

Proof. Suppose that $\xi(p) = 0$. So $\text{rank}(\omega_1(p), \dots, \omega_n(p)) \leq n - 1$, since $a \neq \vec{0}$. However, the n -coframe ω has corank 1, thus $\text{rank}(\omega_1(p), \dots, \omega_n(p)) = n - 1$. That is, $p \in \Sigma^1(\omega)$. Moreover, $\xi(p) = 0$ implies that $T_p M \subset \ker(\xi(p))$ and since $T_p \Sigma^1(\omega) \subset T_p M$, we conclude that p is a zero of $\xi|_{\Sigma^1(\omega)} = 0$. \square

Lemma 3.2. *If $p \in A_{k+1}(\omega)$ then, for each $k = 0, \dots, n - 2$, p is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if p is a zero of $\xi|_{\Sigma^k(\omega)}$.*

Proof. Suppose that $p \in A_{k+1}(\omega)$ and that, locally, we have:

$$\begin{aligned} \mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}; \\ \mathcal{U} \cap \Sigma^{k+1}(\omega) &= \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_{k+1}(x) = 0\}; \end{aligned}$$

for an open neighborhood $\mathcal{U} \subset M$, with $p \in \mathcal{U}$. If p is a zero of the restriction $\xi|_{\Sigma^k(\omega)}$ then $\xi(p) \in N_p^* \Sigma^k(\omega) = \langle dF_1(p), \dots, dF_{m-n+1}(p), d\Delta_2(p), \dots, d\Delta_k(p) \rangle$. In particular, $\xi(p) \in N_p^* \Sigma^{k+1}(\omega)$, therefore p is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$.

On the other hand, if p is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ then $\xi(p) \in N_p^* \Sigma^{k+1}(\omega) \cap \langle \bar{\omega}(p) \rangle$.

Since $p \in A_{k+1}(\omega)$, we have that $p \in \Sigma_{k+1}(\omega) \setminus \Sigma_{k+2}(\omega)$, thus

$$\begin{cases} \dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^k(\omega)) = k; \\ \dim(\langle \bar{\Omega}^{k+1}(p) \rangle \cap N_p^* \Sigma^{k+1}(\omega)) = 0; \end{cases}$$

where $\bar{\Omega}^{k+1}(p)$ represents a smooth basis for a vector subspace supplementary to $\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^k(\omega)$ in $\langle \bar{\omega}(p) \rangle$. Since $\dim(N_p^* \Sigma^k(\omega)) = m - n + k$, $\dim(N_p^* \Sigma^{k+1}(\omega)) = m - n + k + 1$ and $N_p^* \Sigma^k(\omega) \subset N_p^* \Sigma^{k+1}(\omega)$, we have

$$\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{k+1}(\omega)) = \dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^k(\omega)) = k.$$

Thus, $\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^k(\omega) = \langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{k+1}(\omega)$. Therefore, $\xi(p) \in N_p^* \Sigma^k(\omega)$, that is, p is a zero of $\xi|_{\Sigma^k(\omega)}$. \square

Lemma 3.3. *If $p \in A_n(\omega)$ then p is a zero of the restriction $\xi|_{\Sigma^{n-1}(\omega)}$.*

Proof. Analogously to Lemma 3.2, we consider local equations of $\Sigma^n(\omega)$:

$$\mathcal{U} \cap \Sigma^n(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_n(x) = 0\},$$

with $N_x^* \Sigma^n(\omega) = \langle dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_n(x) \rangle$. Since $A_n(\omega) = \Sigma^n(\omega)$, if $p \in A_n(\omega)$ then

$$\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{n-1}(\omega)) = n - 1.$$

Thus, $\langle \bar{\omega}(p) \rangle \subset N_p^* \Sigma^{n-1}(\omega)$ and consequently, $\xi(p) \in N_p^* \Sigma^{n-1}(\omega)$. Therefore, p is a zero of $\xi|_{\Sigma^{n-1}(\omega)}$. \square

Remark 3.1. *If $p \in \Sigma^1(\omega)$ then $\text{rank}(\omega_1(p), \dots, \omega_n(p)) = n - 1$ and, writing $\omega_i = (\omega_i^1, \dots, \omega_i^m)$, we can suppose without loss of generality that*

$$(11) \quad \mathbf{M}(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \dots & \omega_{n-1}^1(x) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \dots & \omega_{n-1}^{n-1}(x) \end{vmatrix} \neq 0,$$

for all x in an open neighborhood $\mathcal{U} \subset M$ with $p \in \mathcal{U}$. In particular, if $p \in \mathcal{U}$ is a singular point of ξ then $a_n \neq 0$, otherwise, we would have $a_1 = \dots = a_{n-1} = a_n = 0$. We will use this fact in next results.

Lemma 3.4. *Let $p \in \Sigma^1(\omega)$ such that $\mathbf{M}(p) \neq 0$. Then $\xi(p) = 0$ if and only if $\sum_{i=1}^n a_i \omega_i^j(p) = 0$, for every $j = 1, \dots, n - 1$.*

Proof. It follows easily from the definition of $\Sigma^1(\omega)$ and ξ . \square

Lemma 3.5. *Let $Z(\xi)$ be the zero set of the 1-form ξ . Then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $Z(\xi) \cap \Sigma^2(\omega) = \emptyset$.*

Proof. Let $\mathcal{U} \subset M$ be an open neighborhood on which $\mathbf{M}(x) \neq 0$ and

$$\mathcal{U} \cap \Sigma^2(V) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = 0\},$$

with $\text{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x)) = m - n + 2$, for each $x \in \Sigma^2(V) \cap \mathcal{U}$.

Let us consider $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \rightarrow \mathbb{R}^{m+1}$ the mapping defined by

$$F(x, a) = (F_1(x), \dots, F_{m-n+1}(x), \Delta_2(x), \sum_{i=1}^n a_i \omega_i^1(x), \dots, \sum_{i=1}^n a_i \omega_i^{n-1}(x)).$$

By Lemma 3.4, if $x \in \Sigma^1(\omega)$ then

$$\sum_{i=1}^n a_i \omega_i(x) = 0 \Leftrightarrow \sum_{i=1}^n a_i \omega_i^j(x) = 0, \forall j = 1, \dots, n-1.$$

Thus, if $(x, a) \in F^{-1}(\vec{0})$ we have that $x \in Z(\xi) \cap \Sigma^2(V)$. Furthermore, the Jacobian matrix of F at a point $(x, a) \in F^{-1}(\vec{0})$:

$$\begin{bmatrix} dF_1(x) & \vdots & & & & \\ \vdots & \vdots & & & & \\ dF_{m-n+1}(x) & \vdots & & & & \\ d\Delta_2(x) & \vdots & & & & \\ \dots & \vdots & \dots & \dots & \dots & \dots \\ & \vdots & \omega_1^1(x) & \dots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ (*) & \vdots & \omega_1^2(x) & \dots & \omega_{n-1}^2(x) & \omega_n^2(x) \\ & \vdots & \vdots & \ddots & \vdots & \vdots \\ & \vdots & \omega_1^{n-1}(x) & \dots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \end{bmatrix}$$

has rank $m+1$. That is, $\vec{0}$ is regular value of F and $F^{-1}(\vec{0})$ is a submanifold of dimension $n-1$. Let $\pi : F^{-1}(\vec{0}) \rightarrow \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \setminus \{\vec{0}\}$ given by $\pi(x, a) = a$, by Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. However, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x, a) \in \mathcal{U} \times \{a\} : x \in Z(\xi) \cap \Sigma^2(\omega)\}$. Thus, $Z(\xi) \cap \Sigma^2(\omega) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. \square

Lemma 3.6. *Let $Z(\xi|_{\Sigma^k(\omega)})$ be the zero set of the restriction of the 1-form ξ to $\Sigma^k(\omega)$, with $k \geq 1$. Then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega) = \emptyset$.*

Proof. For each $k = 1, \dots, n-2$, let $\mathcal{U} \subset M$ be an open neighborhood on which,

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+k}(x) = 0\},$$

with $\text{rank}(dF_1(x), \dots, dF_{m-n+k}(x)) = m-n+k$, for all $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and

$$\mathcal{U} \cap \Sigma^{k+2}(V) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+k+2}(x) = 0\},$$

with $\text{rank}(dF_1(x), \dots, dF_{m-n+k+2}(x)) = m-n+k+2$, for all $x \in \mathcal{U} \cap \Sigma^{k+2}(V)$.

By Szafraniec's characterization (see [16, p. 196]) adapted to 1-forms, x is a zero of the restriction $\xi|_{\Sigma^k(\omega)}$ if and only if there exists $(\lambda_1, \dots, \lambda_{m-n+k}) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=1}^{m-n+k} \lambda_j dF_j(x).$$

Let us write $\xi(x) = (\xi_1(x), \dots, \xi_m(x))$, where $\xi_s(x) = \sum_{i=1}^n a_i \omega_i^s(x)$, $s = 1, \dots, m$, we define

$$N_s(x, a, \lambda) := \xi_s(x) - \sum_{j=1}^{m-n+k} \lambda_j \frac{\partial F_j}{\partial x_s}(x),$$

so that $\xi|_{\Sigma^k(\omega)}(x) = 0$ if and only if $N_s(x, a, \lambda) = 0$, for all $s = 1, \dots, m$.

Let $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \rightarrow \mathbb{R}^{2m-n+k+2}$ be the mapping defined by

$$F(x, a, \lambda) = (F_1, \dots, F_{m-n+k+2}, N_1, \dots, N_m),$$

if $(x, a, \lambda) \in F^{-1}(\vec{0})$ then $x \in Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega)$ and the Jacobian matrix of F at (x, a, λ) :

$$\begin{bmatrix} dF_1(x) & \vdots & & & \\ \vdots & \vdots & & & \\ dF_{m-n+k+2}(x) & \vdots & & & \\ \dots & \dots & \dots & \dots & \dots \\ d_x N_1(x, a, \lambda) & \vdots & & \vdots & \\ \vdots & \vdots & B_{m \times n} & \vdots & C_{m \times (m-n+k)} \\ d_x N_m(x, a, \lambda) & \vdots & & \vdots & \end{bmatrix}$$

has rank $2m - n + k + 1$, where $O_{(m-n+k+2) \times (m+k)}$ is a null matrix, $B_{m \times n}$ is a matrix whose columns vectors are given by the coefficients of the 1-forms $\omega_1(x), \dots, \omega_n(x)$ of the n -coframe ω :

$$B_{m \times n} = \begin{bmatrix} \omega_1^1(x) & \dots & \omega_n^1(x) \\ \vdots & \ddots & \vdots \\ \omega_1^m(x) & \dots & \omega_n^m(x) \end{bmatrix}$$

and $C_{m \times (m-n+k)}$ is the matrix whose columns vectors are, up to sign, the coefficients of the derivatives dF_1, \dots, dF_{m-n+k} with respect to x :

$$C_{m \times (m-n+k)} = \begin{bmatrix} -\frac{\partial F_1}{\partial x_1}(x) & \dots & -\frac{\partial F_{m-n+k}}{\partial x_1}(x) \\ \vdots & \ddots & \vdots \\ -\frac{\partial F_1}{\partial x_m}(x) & \dots & -\frac{\partial F_{m-n+k}}{\partial x_m}(x) \end{bmatrix}.$$

Note that, if $(x, a, \lambda) \in F^{-1}(\vec{0})$ then, in particular, $x \in \Sigma^{k+1}(\omega)$ and by Lemma 2.6, $\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = k$. Thus, $\dim(\langle \bar{\omega}(x) \rangle + N_x^* \Sigma^k(\omega)) = m - 1$. Therefore,

$$\text{rank} \begin{bmatrix} B_{m \times n} & \vdots & C_{m \times (m-n+k)} \end{bmatrix} = m - 1$$

and the Jacobian matrix of F at (x, a, λ) has rank $2m - n + k + 1$. That is, $F^{-1}(\vec{0})$ has dimension less or equal to $n - 1$. Let $\pi : F^{-1}(\vec{0}) \rightarrow \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \setminus \{\vec{0}\}$, that is, $\pi(x, a, \lambda) = a$. By Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. However,

$$\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x, a, \lambda) \in \mathcal{U} \times \{a\} \times \mathbb{R}^{m-n+k} \mid x \in Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega)\}.$$

Thus, $Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. \square

4. NON-DEGENERATE ZEROS OF A GENERIC 1-FORM $\xi(x)$ ASSOCIATED TO A MORIN n -COFRAME

In this section we will verify that, generically, the 1-form $\xi(x)$ and its restrictions $\xi|_{\Sigma^k(\omega)}$, $\xi|_{A_k(\omega)}$ admit only non-degenerate zeros. Furthermore, we will see how these non-degenerate zeros can be related. We start with some technical lemmas.

Lemma 4.1. *Let A be a square matrix of order m given by:*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

If there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \setminus \{\vec{0}\}$ such that $\sum_{j=1}^m \lambda_j a_{ij} = 0$, $i = 1, \dots, m$, then

$$\lambda_j \operatorname{cof}(a_{ik}) - \lambda_k \operatorname{cof}(a_{ij}) = 0, \forall j, k = 1, \dots, m.$$

Lemma 4.2. *Let us consider the matrix*

$$M_i(x) = \begin{bmatrix} \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{bmatrix}.$$

If x is a zero of ξ then for $\ell \in \{1, \dots, n-1\}$, $j \in \{1, \dots, n-1, i\}$ and $i \in \{n, \dots, m\}$, we have

$$a_n \operatorname{cof}(\omega_\ell^j, M_i) = a_\ell \operatorname{cof}(\omega_n^j, M_i).$$

Proof. This result is a consequence of Lemma 4.1 applied to the matrix $A = M_i(x)$, where $a_{\ell j} = \omega_j^\ell(x)$, for $j = 1, \dots, n$ and $\ell = 1, \dots, n-1, i$. It is enough to take $(\lambda_1, \dots, \lambda_n) = (a_1, \dots, a_n)$. \square

Lemma 4.3. *Let $\mathcal{U} \subset \mathbb{R}^m$ be an open set and let $H : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \rightarrow \mathbb{R}^m$ be a smooth mapping given by $H(x, a) = (h_1(x, a), \dots, h_m(x, a))$. If*

$$\operatorname{rank}(dh_1(x, a), \dots, dh_m(x, a)) = m, \forall (x, a) \in H^{-1}(\vec{0})$$

then $\operatorname{rank}(d_x h_1(x, a), \dots, d_x h_m(x, a)) = m$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$.

In the previous section we proved that every zero of ξ belongs to $\Sigma^1(\omega)$. Next, we will show that, generically, such zeros belong to $A_1(\omega)$ and they are non-degenerate. To do this, we must find explicit equations that define locally the manifolds $T^*M^{n, n-1}$ and $\Sigma^1(\omega)$.

Lemma 4.4. *Let $(p, \tilde{\varphi}) \in T^*M^{n, n-1}$, it is possible to exhibit, explicitly, functions $m_i(x, \varphi) : \tilde{\mathcal{U}} \rightarrow \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\tilde{\mathcal{U}} \subset T^*M^n$, with $(p, \tilde{\varphi}) \in \tilde{\mathcal{U}}$, such that, locally*

$$T^*M^{n, n-1} = \left\{ (x, \varphi) \in \tilde{\mathcal{U}} \mid m_n = \dots = m_m = 0 \right\}$$

*with $\operatorname{rank}(dm_n, \dots, dm_m) = m - n + 1$, for all $(x, \varphi) \in T^*M^{n, n-1} \cap \tilde{\mathcal{U}}$.*

Proof. Let $(p, \tilde{\varphi}) \in T^*M^{n, n-1}$, we may suppose without loss of generality that

$$m(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} \end{vmatrix} \neq 0$$

for (x, φ) in an open neighborhood $\tilde{\mathcal{U}}$ of T^*M^n , with $(p, \tilde{\varphi}) \in \tilde{\mathcal{U}}$. In this situation, $T^*M^{n, n-1}$ can be locally defined as

$$T^*M^{n, n-1} = \left\{ (x, \varphi) \in \tilde{\mathcal{U}} \mid m_n = \dots = m_m = 0 \right\}$$

where $m_i := m_i(\varphi)$ is the determinant

$$m_i(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^i & \varphi_2^i & \cdots & \varphi_{n-1}^i & \varphi_n^i \end{vmatrix}, \quad i = n, \dots, m.$$

Let us verify that $\text{rank}(dm_n, \dots, dm_m) = m - n + 1$ in $(T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$.

For clearer notations, consider $I = \{1, \dots, n\}$ and $I_i = \{1, \dots, n-1, i\}$ for each $i \in \{n, \dots, m\}$. Then

$$(12) \quad dm_i(\varphi) = \sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) d\varphi_j^\ell,$$

where $\text{cof}(\varphi_j^\ell, m_i)$ is the cofactor of φ_j^ℓ in the matrix

$$\begin{bmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^i & \varphi_2^i & \cdots & \varphi_{n-1}^i & \varphi_n^i \end{bmatrix}$$

and

$$d\varphi_j^\ell = \left(\frac{\partial \varphi_j^\ell}{\partial \varphi_1^1}, \dots, \frac{\partial \varphi_j^\ell}{\partial \varphi_1^m}, \frac{\partial \varphi_j^\ell}{\partial \varphi_2^1}, \dots, \frac{\partial \varphi_j^\ell}{\partial \varphi_2^m}, \dots, \frac{\partial \varphi_j^\ell}{\partial \varphi_n^1}, \dots, \frac{\partial \varphi_j^\ell}{\partial \varphi_n^m} \right)$$

is the vector whose coordinate at the position $(j-1)m + \ell$ is equal to 1 and all the others are zero. In particular, since $i \in \{n, \dots, m\}$,

$$d\varphi_n^i = (0, \dots, 0, \underbrace{0, \dots, 0}_{m-n+1}, \underbrace{1, \dots, 0}_{n \text{ times}}) \in (\mathbb{R}^m)^* \times \dots \times (\mathbb{R}^m)^*$$

and the $m - n + 1$ last coordinates of $d\varphi_j^\ell$ are zero for all $j \neq n$ or $\ell \neq i$. Moreover, $\text{cof}(\varphi_n^i, m_i) = m(\varphi) \neq 0$, for $i = n, \dots, m$. Thus,

$$\frac{\partial(m_n, \dots, m_m)}{\partial(\varphi_n^1, \dots, \varphi_n^m)} = \begin{vmatrix} \text{cof}(\varphi_n^1, m_n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{cof}(\varphi_n^m, m_m) \end{vmatrix}.$$

That is, for all $(x, \varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$, we have

$$(13) \quad \frac{\partial(m_n, \dots, m_m)}{\partial(\varphi_n^1, \dots, \varphi_n^m)} = m(\varphi)^{(m-n+1)} \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \neq 0.$$

Therefore, $\text{rank}(m_n, \dots, m_m) = m - n + 1$ for all $(x, \varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$. \square

Lemma 4.5. *Let $p \in \Sigma^1(\omega)$ be a singular point of ω , it is possible to exhibit, explicitly, functions $\mathbf{M}_i(x) : \mathcal{U} \rightarrow \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\mathcal{U} \subset M$, with $p \in \mathcal{U}$, such that, locally*

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$$

with $\text{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$, for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$.

Proof. Let $\omega = (\omega_1, \dots, \omega_n)$ be a Morin n -coframe and let $p \in \Sigma^1(\omega)$. By Remark 3.1, we can consider $\mathcal{U} \subset M$ an open neighborhood with $p \in \mathcal{U}$, where $\mathbf{M}(x) \neq 0$. Thus, in this neighborhood the set $\Sigma^1(\omega)$ can be defined as

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n = \dots = \mathbf{M}_m = 0\},$$

where $\mathbf{M}_i := \mathbf{M}_i(x)$ is the determinant

$$(14) \quad \tilde{\mathbf{M}}_i(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \omega_2^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{vmatrix}$$

for $i = n, \dots, m$. Let us verify that $\text{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$, for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$.

Let $G(\omega) = \{(x, \omega_1(x), \dots, \omega_n(x)) \mid x \in M\}$ be the graph of the n -coframe ω . By Lemma 4.4, we can consider an open neighborhood $\tilde{\mathcal{U}}$ in T^*M^n with $(p, \omega(p)) \in \tilde{\mathcal{U}}$ and $\pi_x(\tilde{\mathcal{U}}) = \mathcal{U}$, where $\pi_x : (\mathbb{R}^m)^n \rightarrow \mathbb{R}^m$ is the projection on the m first coordinates, so that the manifolds $T^*M^{n,n-1}$ and $G(\omega)$ can be locally defined as:

$$T^*M^{n,n-1} = \{(x, \varphi) \in \tilde{\mathcal{U}} \mid m_n = \dots = m_m = 0\},$$

with $\text{rank}(dm_n, \dots, dm_m) = m - n + 1$ on $T^*M^{n,n-1} \cap \tilde{\mathcal{U}}$; and

$$\begin{aligned} G(\omega) &= \{(x, \varphi) \in \tilde{\mathcal{U}} \mid \varphi_j^\ell = \omega_j^\ell(x); j = 1, \dots, n; \ell = 1, \dots, m\} \\ &= \{(x, \varphi) \in \tilde{\mathcal{U}} \mid g_{\ell j}(x, \varphi) = 0; j = 1, \dots, n; \ell = 1, \dots, m\}, \end{aligned}$$

with $\text{rank}(dg_{11}, \dots, dg_{m1}, \dots, dg_{1n}, \dots, dg_{mn}) = nm$ on $G(\omega) \cap \tilde{\mathcal{U}}$, where the functions $g_{\ell j} : T^*M^n \rightarrow \mathbb{R}$ are given by $g_{\ell j}(x, \varphi) = \varphi_j^\ell - \omega_j^\ell(x)$.

Let $x \in \Sigma^1(\omega) \cap \mathcal{U}$, then $G(\omega) \cap T^*M^{n,n-1}$ at $(x, \omega(x)) \in \tilde{\mathcal{U}}$ and, at this point, $\text{rank}(dm_n, \dots, dm_m, dg_{11}, \dots, dg_{mn}) = m - n + 1 + nm$. That is, the matrix

$$(15) \quad \begin{bmatrix} dm_n \\ \vdots \\ dm_m \\ dg_{11} \\ \vdots \\ dg_{mn} \end{bmatrix} = \begin{bmatrix} & \vdots & d_\varphi m_n \\ & \vdots & \vdots \\ O_{(m-n+1) \times m} & \vdots & \vdots \\ & \vdots & d_\varphi m_m \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ & -d_x \omega_1^1 & \vdots & & & \\ & \vdots & \vdots & & Id_{(nm)} & \\ & -d_x \omega_n^m & \vdots & & & \end{bmatrix}$$

has maximal rank at $(x, \omega(x))$, where $O_{(m-n+1) \times m}$ is the null matrix of size $(m-n+1) \times m$, $Id_{(nm)}$ represents the identity matrix of order nm and d_x and d_φ denote the derivatives with respect to $x = (x_1, \dots, x_m)$ and $\varphi = (\varphi_1^1, \dots, \varphi_1^m, \dots, \varphi_n^1, \dots, \varphi_n^m)$ respectively.

We have that Equation (12) of Lemma 4.4 is the derivative of m_i with respect to φ . Thus, we can write

$$(16) \quad dm_i(\varphi) = \sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) f_{\ell j}$$

for $I = \{1, \dots, n\}$, $I_i = \{1, \dots, n-1, i\}$ and $i = n, \dots, m$. Where $f_{\ell j}$ denotes the vector $f_{\ell j} = (0, \dots, 0, 0, \dots, 0, 1, 0, \dots, 0) \in \underbrace{(\mathbb{R}^m)^* \times \dots \times (\mathbb{R}^m)^*}_{n+1 \text{ times}}$ whose coordinates

are all zero, except at the position $jm + \ell$, for each $j \in I$ and $\ell \in I_i$.

We also have,

$$d\mathbf{M}_i(x) = \sum_{j \in I, \ell \in I_i} \text{cof}(\omega_j^\ell(x), \mathbf{M}_i(x)) d_x \omega_j^\ell(x)$$

and since $(x, \omega(x)) \in G(\omega) \cap T^*M^{n, n-1}$, we have $\omega_j^\ell(x) = \varphi_j^\ell$ so that

$$(17) \quad d\mathbf{M}_i(x) = \sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) d_x \omega_j^\ell(x).$$

Let us suppose that $\text{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) < m - n + 1$. Then, there exists $(\alpha_n, \dots, \alpha_m) \neq (0, \dots, 0)$ such that

$$\sum_{i=n}^m \alpha_i d\mathbf{M}_i(x) = 0.$$

Thus,

$$(18) \quad 0 = \sum_{i=n}^m \alpha_i d\mathbf{M}_i(x) \stackrel{(17)}{=} \sum_{i=n}^m \alpha_i \left[\sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) d_x \omega_j^\ell(x) \right].$$

Let $d\tilde{\omega}_j^\ell(x) = (d_x \omega_j^\ell(x), 0, \dots, 0) \in \underbrace{(\mathbb{R}^m)^* \times \dots \times (\mathbb{R}^m)^*}_{n+1 \text{ times}}$, we have

$$(19) \quad \begin{aligned} \sum_{i=n}^m \alpha_i \left[\sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) dg_{\ell j} \right] &\stackrel{(15)}{=} \sum_{i=n}^m \alpha_i \left[\sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) (f_{\ell j} - d\tilde{\omega}_j^\ell) \right] \\ &\stackrel{(18)}{=} \sum_{i=n}^m \alpha_i \left[\sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) f_{\ell j} \right] \\ &\stackrel{(16)}{=} \sum_{i=n}^m \alpha_i dm_i. \end{aligned}$$

On the other hand,

$$(20) \quad \sum_{i=n}^m \alpha_i \left[\sum_{j \in I, \ell \in I_i} \text{cof}(\varphi_j^\ell, m_i) dg_{\ell j} \right] = \sum_{j \in I, \ell \in \{1, \dots, m\}} \beta_{\ell j} dg_{\ell j}$$

where

$$\beta_{\ell j} = \begin{cases} \sum_{i=n}^m \alpha_i \text{cof}(\varphi_j^\ell, m_i), & j \in I, \ell = 1, \dots, n-1; \\ \alpha_\ell \text{cof}(\varphi_j^\ell, m_\ell), & j \in I, \ell = n, \dots, m. \end{cases}$$

Since $(\alpha_n, \dots, \alpha_m) \neq (0, \dots, 0)$, by Equations (19) and (20), we obtain

$$\sum_{j, \ell} \beta_{\ell j} dg_{\ell j} - \sum_{i=n}^m \alpha_i dm_i = 0$$

which is a linear combination (with non-zero coefficients) of the row vectors of the matrix (15). This is a contradiction, since $\text{rank}(dm_n, \dots, dm_m, dg_{11}, \dots, dg_{mn})$ is maximal. Therefore $\text{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$. \square

Lemma 4.6. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the 1-form $\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$ admits only non-degenerate zeros. Moreover, such zeros belong to $A_1(\omega)$.*

Proof. Suppose that $p \in M$ is a zero of ξ . Then, by Lemmas 3.1 and 3.5, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$ we have that $p \in \Sigma^1(\omega) \setminus \Sigma^2(\omega)$, that is, $p \in A_1(\omega)$. Assume that $\mathbf{M}(x) \neq 0$ in an open neighborhood $\mathcal{U} \subset M$ of p (see Remark 3.1) so that $\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$. Let us write

$$\xi_s(x) = \sum_{i=1}^n a_i \omega_i^s(x), \quad s = 1, \dots, m$$

and let us consider the mapping $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \rightarrow \mathbb{R}^m$ defined by

$$F(x, a) = (\mathbf{M}_n(x), \dots, \mathbf{M}_m(x), \xi_1(x), \dots, \xi_{n-1}(x)).$$

Its Jacobian matrix at a point (x, a) is given by:

$$\text{Jac } F(x, a) = \begin{bmatrix} d_x \mathbf{M}_n(x) & \vdots & & & & \\ \vdots & \vdots & & & & \\ & & O_{(m-n) \times n} & & & \\ d_x \mathbf{M}_m(x) & \vdots & & & & \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_x \xi_1(x) & \vdots & \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_x \xi_{n-1}(x) & \vdots & \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \end{bmatrix}.$$

Note that, by Lemma 3.4, $F^{-1}(\vec{0})$ corresponds to the zeros of ξ on $\Sigma^1(\omega) \cap \mathcal{U}$. Since $\mathbf{M}(x) \neq 0$ and $\text{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$ for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$, then $\text{rank}(\text{Jac } F(x, a)) = m$ for all $(x, a) \in F^{-1}(\vec{0})$. Thus, $\dim F^{-1}(\vec{0}) = n$.

Let $\pi : F^{-1}(\vec{0}) \rightarrow \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection $\pi(x, a) = a$, by Sard's Theorem, almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$ is a regular value of π and $\dim(\pi^{-1}(a) \cap F^{-1}(\vec{0})) = 0$. That is, for almost every a , the zeros of ξ are isolated in $\Sigma^1(\omega)$. Let us proof that, moreover, these zeros are non-degenerate.

Since $\text{rank}(\text{Jac } F(x, a)) = m$, for all $(x, a) \in F^{-1}(\vec{0})$, then by Lemma 4.3 we have that $\text{rank}(d_x \mathbf{M}_n(p), \dots, d_x \mathbf{M}_m(p), d_x \xi_1(p), \dots, d_x \xi_{n-1}(p)) = m$, which happens if and only if $\text{rank}(B) = m$, where B is the matrix

$$B = \begin{bmatrix} d_x \xi_1(p) \\ \vdots \\ d_x \xi_{n-1}(p) \\ a_n d_x \mathbf{M}_n(p) \\ \vdots \\ a_n d_x \mathbf{M}_m(p) \end{bmatrix}$$

whose row vectors we will denote by $R_i, i = 1, \dots, m$ (by Remark 3.1, $a_n \neq 0$).

Let us denote $I = \{1, \dots, n\}$ and $I_i = \{1, \dots, n-1, i\}$ for each $i \in \{n, \dots, m\}$. By Equation (14), we can write

$$d\mathbf{M}_i(x) = \sum_{\ell \in I, j \in I_i} \text{cof}(\omega_\ell^j(x), M_i) d\omega_\ell^j(x)$$

and by Lemma 4.2,

$$d\mathbf{M}_i(p) = \sum_{\ell \in I, j \in I_i} \frac{a_\ell}{a_n} \text{cof}(\omega_\ell^j(p), M_i) d\omega_\ell^j(p).$$

Thus,

$$\begin{aligned}
a_n d\mathbf{M}_i(p) &= \sum_{\ell \in I, j \in I_i} a_\ell \operatorname{cof}(\omega_n^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{j \in I_i} \operatorname{cof}(\omega_n^j(p), M_i) \left[\sum_{\ell \in I} a_\ell d\omega_\ell^j(p) \right] \\
&= \sum_{j \in I_i} \operatorname{cof}(\omega_n^j(p), M_i) [d_x \xi_j(p)] \\
&= \operatorname{cof}(\omega_n^i(p), M_i) [d_x \xi_i(p)] + \sum_{j \in I_i \setminus \{i\}} \operatorname{cof}(\omega_n^j(p), M_i) [d_x \xi_j(p)].
\end{aligned}$$

Note that, $\operatorname{cof}(\omega_n^i(p), M_i) = \mathbf{M}(p) \neq 0$, for all $i = n, \dots, m$. Then, for each $i = n, \dots, m$, we replace the i^{th} row R_i of matrix B by

$$\frac{1}{\operatorname{cof}(\omega_n^i(p), M_i)} \left(R_i - \sum_{j=1}^{n-1} \operatorname{cof}(\omega_n^j(p), M_i) R_j \right)$$

so that we obtain the matrix of maximal rank:

$$\begin{bmatrix} d_x \xi_1(p) \\ \vdots \\ d_x \xi_{n-1}(p) \\ d_x \xi_n(p) \\ \vdots \\ d_x \xi_m(p) \end{bmatrix}.$$

Therefore, the zeros of $\xi(x)$ are non-degenerate. \square

Lemma 4.7. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the 1-form $\xi|_{A_k(\omega)}$ admits only non-degenerate zeros, $k \geq 2$.*

Proof. Suppose that $\xi|_{A_k(\omega)}(p) = 0$. By Lemmas 2.7 and 4.5, we can consider $\mathcal{U} \subset M$ an open neighborhood of p where $\mathbf{M}(x) \neq 0$ and on which the respective singular sets ($k = 2, \dots, n$) can be locally defined as

$$\begin{aligned}
\mathcal{U} \cap \Sigma^1(\omega) &= \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}, \\
\mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\},
\end{aligned}$$

with

$$\begin{aligned}
\operatorname{rank}(d\mathbf{M}_n, \dots, d\mathbf{M}_m) &= m - n + 1, \forall x \in \Sigma^1(\omega) \cap \mathcal{U}, \\
\operatorname{rank}(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k) &= m - n + k, \forall x \in \Sigma^k(\omega) \cap \mathcal{U}.
\end{aligned}$$

Analogously to the proof of Lemma 3.6, by Szafraniec's characterization (see [16, p. 196]), x is a zero of the restriction $\xi|_{\Sigma^k(\omega)}$ if and only if there exists $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_k) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=n}^m \lambda_j d\mathbf{M}_j(x) + \sum_{\ell=2}^k \beta_\ell d\Delta_\ell(x).$$

Let us consider the functions

$$N_s(x, a, \lambda, \beta) := \xi_s(x) - \sum_{j=n}^m \lambda_j \frac{\partial \mathbf{M}_j}{\partial x_s}(x) - \sum_{\ell=2}^k \beta_\ell \frac{\partial \Delta_\ell}{\partial x_s}(x), \quad s = 1, \dots, m,$$

so that $\xi|_{\Sigma^k(\omega)}(x) = 0$ if and only if $N_s(x, a, \lambda, \beta) = 0$, for all $s = 1, \dots, m$.

Let $G : \mathcal{U} \setminus \{\Delta_{k+1} = 0\} \times \mathbb{R}^n \setminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \rightarrow \mathbb{R}^{2m-n+k}$ be the mapping

$$G(x, a, \lambda, \beta) = (\mathbf{M}_n, \dots, \mathbf{M}_m, \Delta_2, \dots, \Delta_k, N_1, \dots, N_m).$$

Its Jacobian matrix at a point $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$ is given by:

$$\text{Jac } G(x, a, \lambda, \beta) = \begin{bmatrix} d_x \mathbf{M}_n(x) & \vdots & & & \\ \vdots & \vdots & & & \\ d_x \mathbf{M}_m(x) & \vdots & & & \\ d_x \Delta_2(x) & \vdots & O_{(m-n+k) \times (m+k)} & & \\ \vdots & \vdots & & & \\ d_x \Delta_k(x) & \vdots & & & \\ \dots & \dots & \dots & \dots & \dots \\ d_x N_1(x, a, \lambda, \beta) & \vdots & \vdots & & \\ \vdots & \vdots & B_{m \times n} & \vdots & C_{m \times (m-n+k)} \\ d_x N_m(x, a, \lambda, \beta) & \vdots & \vdots & & \end{bmatrix}$$

where $O_{(m-n+k) \times (m+k)}$ is a null matrix, $B_{m \times n}$ is the matrix whose column vectors are given by the coefficients of the 1-forms $\omega_1(x), \dots, \omega_n(x)$ and $C_{m \times (m-n+k)}$ is the matrix whose column vectors are, up to sign, the coefficients of the derivatives $d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k$ with respect to x :

$$C_{m \times (m-n+k)} = \begin{bmatrix} -\frac{\partial \mathbf{M}_n}{\partial x_1}(x) & \dots & -\frac{\partial \mathbf{M}_m}{\partial x_1}(x) & -\frac{\partial \Delta_2}{\partial x_1}(x) & \dots & -\frac{\partial \Delta_k}{\partial x_1}(x) \\ \vdots & \ddots & \vdots & \vdots & & \\ -\frac{\partial \mathbf{M}_n}{\partial x_m}(x) & \dots & -\frac{\partial \mathbf{M}_m}{\partial x_m}(x) & -\frac{\partial \Delta_2}{\partial x_m}(x) & \dots & -\frac{\partial \Delta_k}{\partial x_m}(x) \end{bmatrix}.$$

Thus, if $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$ then $x \in \Sigma^k(\omega) \cap \mathcal{U}$, $\Delta_{k+1}(x) \neq 0$ and $\xi_{|\Sigma^k(\omega)}(x) = 0$. And since $A_k(\omega) = \Sigma^k(\omega) \setminus \Sigma^{k+1}(\omega)$, we have $x \in A_k(\omega) \cap Z(\xi_{|\Sigma^k(\omega)})$, for all $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$.

On the other hand, if $x \in A_k(\omega)$ then $\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = k-1$ and $\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = k-1$, so that $\dim(\langle \bar{\omega}(x) \rangle + N_x^* \Sigma^k(\omega)) = m$. Thus,

$$\text{rank} \begin{bmatrix} d_x N_1(x, a, \lambda, \beta) & \vdots & & \vdots \\ \vdots & \vdots & B_{m \times n} & \vdots \\ d_x N_m(x, a, \lambda, \beta) & \vdots & & \vdots \end{bmatrix} = m$$

and the Jacobian matrix of G has maximal rank at every $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$. Therefore, $\dim G^{-1}(\vec{0}) = (2m+k) - (2m-n+k) = n$. Let $\pi : G^{-1}(\vec{0}) \rightarrow \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection $\pi(x, a, \lambda, \beta) = a$, then almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$ is a regular value of π . So, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\dim(\pi^{-1}(a) \cap G^{-1}(\vec{0})) = 0$ and $\pi^{-1}(a) \pitchfork G^{-1}(\vec{0})$. Therefore, the zeros of $\xi_{|A_k(\omega)}$ are non-degenerate. \square

Lemma 4.8. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the 1-form $\xi_{|A_1(\omega)}$ admits only non-degenerate zeros.*

Proof. This proof follows analogously the proof of Lemma 4.7. \square

By Lemma 3.2, if $p \in A_{k+1}(\omega)$, then p is a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if p is a zero of $\xi|_{\Sigma^k(\omega)}$. The next results state that this relation also holds for non-degenerate zeros.

Lemma 4.9. *Let $p \in A_1(\omega)$ be a zero of $\xi|_{\Sigma^1(\omega)}$, then p is a non-degenerate zero of $\xi|_{\Sigma^1(\omega)}$ if and only if p is a non-degenerate zero of ξ .*

Proof. Let $p \in A_1(\omega)$ be a zero of the restriction $\xi|_{\Sigma^1(\omega)}$ and let $\mathcal{U} \subset M$ be an open neighborhood of p at which $\mathbf{M}(x) \neq 0$ and $\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$. By Szafraniec's characterization ([16, p. 196]), $\exists!(\lambda_n, \dots, \lambda_m) \in \mathbb{R}^{m-n+1}$, such that

$$\xi(p) + \sum_{i=n}^m \lambda_i d\mathbf{M}_i(p) = 0.$$

Furthermore, p is a non-degenerate zero of $\xi|_{\Sigma^1(\omega)}$ if and only if the matrix

$$(21) \quad \begin{bmatrix} & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\ \text{Jac} \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i \right) (p) & \vdots & \vdots & \ddots & \vdots \\ & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\ \dots & \dots & \dots & \dots & \dots \\ & d_x \mathbf{M}_n(p) & \vdots & & \\ & \vdots & \vdots & & O_{(m-n+1)} \\ & d_x \mathbf{M}_m(p) & \vdots & & \end{bmatrix}$$

is non-singular. Since $\xi(p) = 0$, then $p \in \Sigma^1(\omega) \cap \mathcal{U}$ and $\lambda_n d\mathbf{M}_n(p) + \dots + \lambda_m d\mathbf{M}_m(p) = \vec{0}$, thus, $\lambda_n = \dots = \lambda_m = 0$ and, writing $\xi = (\xi_1, \dots, \xi_m)$, we have

$$\text{Jac} \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i \right) (p) = \begin{bmatrix} d_x \xi_1(p) \\ \vdots \\ d_x \xi_m(p) \end{bmatrix}.$$

This means that the Matrix (21) is non-singular if and only if the matrix

$$(22) \quad \begin{bmatrix} d_x \xi_1(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_x \xi_m(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\ \dots & \dots & \dots & \dots & \dots \\ a_n d_x \mathbf{M}_n(p) & \vdots & & & \\ \vdots & \vdots & & & O_{(m-n+1)} \\ a_n d_x \mathbf{M}_m(p) & \vdots & & & \end{bmatrix}$$

is non-singular (by Remark 3.1, $a_n \neq 0$). By Equation (14),

$$d_x \mathbf{M}_i(x) = \sum_{\ell \in I, j \in I_i} \text{cof}(\omega_\ell^j(x), M_i) d\omega_\ell^j(x),$$

and applying Lemma 4.2, we obtain

$$\begin{aligned}
a_n d_x \mathbf{M}_i(p) &= \sum_{\ell \in I, j \in I_i} a_\ell \operatorname{cof}(\omega_n^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{j \in I_i} \operatorname{cof}(\omega_n^j(p), M_i) \left[\sum_{\ell \in I} a_\ell d\omega_\ell^j(p) \right] \\
&= \sum_{j \in I_i} \operatorname{cof}(\omega_n^j(p), M_i) [d_x \xi_j(p)].
\end{aligned}$$

Let us denote the m first row vectors of Matrix (22) by $L_j, j = 1, \dots, m$, and let us denote the $m - n + 1$ last row vectors of Matrix (22) by $R_i, i = n, \dots, m$:

$$\begin{aligned}
L_j &= \left(d_x \xi_j(p), \frac{\partial \mathbf{M}_n}{\partial x_j}(p), \dots, \frac{\partial \mathbf{M}_m}{\partial x_j}(p) \right); \\
R_i &= \left(a_n \frac{\partial \mathbf{M}_i}{\partial x_1}(p), \dots, a_n \frac{\partial \mathbf{M}_i}{\partial x_m}(p), \vec{0} \right).
\end{aligned}$$

We replace each row vector $R_i, i = n, \dots, m$, by $R_i - \sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) L_j$ so that we obtain

$$R_i = \left(\underbrace{0, \dots, 0}_{m \text{ times}}, - \sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_n}{\partial x_j}, \dots, - \sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_m}{\partial x_j} \right)$$

and the Matrix (22) becomes:

$$(23) \quad \begin{bmatrix} d_x \xi_1(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_x \xi_m(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\ \dots & \dots & \dots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O_{(m-n+1) \times m} & \vdots & \mathbf{M}'_{(m-n+1)} & \vdots & \vdots \end{bmatrix}$$

where

$$\mathbf{M}'_{(m-n+1)} = - (m_{ij})_{n \leq i, j \leq m}$$

is the matrix given by

$$(24) \quad m_{ij} = \sum_{k \in I_i} \operatorname{cof}(\omega_n^k, M_i) \frac{\partial \mathbf{M}_j}{\partial x_k}, \quad i, j = n, \dots, m.$$

Next, we will verify that the matrix \mathbf{M}' is non-singular. Since $p \in A_1(\omega)$, then $\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^1(\omega)) = 0$ and $\dim(\langle \bar{\omega}(p) \rangle \oplus N_p^* \Sigma^1(\omega)) = m$. Since $\mathbf{M}(p) \neq 0$,

$\{\omega_1(p), \dots, \omega_{n-1}(p)\}$ is a basis of the space $\langle \bar{\omega}(p) \rangle$. Thus the matrix

$$(25) \quad \begin{bmatrix} \omega_1^1(p) & \cdots & \omega_1^{n-1}(p) & \omega_1^n(p) & \cdots & \omega_1^m(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1}^1(p) & \cdots & \omega_{n-1}^{n-1}(p) & \omega_{n-1}^n(p) & \cdots & \omega_{n-1}^m(p) \\ \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_{n-1}}(p) & \frac{\partial \mathbf{M}_n}{\partial x_n}(p) & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{M}_m}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_{n-1}}(p) & \frac{\partial \mathbf{M}_m}{\partial x_n}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \end{bmatrix}$$

has rank maximal. Let us denote the row vectors of Matrix (25) by $L'_j, j = 1, \dots, m$. For $j = 1, \dots, n-1$, we replace L'_j by

$$(26) \quad \sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) L'_k = \left(\sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^1, \dots, \sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^m \right),$$

where

$$\sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^\ell = \begin{cases} \mathbf{M}, & \ell = j; \\ 0 & \ell = 1, \dots, n-1 \text{ and } \ell \neq j; \\ -\text{cof}(\omega_n^j, \mathbf{M}_\ell), & \ell = n, \dots, m. \end{cases}$$

Indeed,

- For $\ell = 1, \dots, n-1$ with $\ell = j$, we have:

$$\sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^j = \begin{vmatrix} \omega_1^1 & \cdots & \omega_k^1 & \cdots & \omega_{n-1}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \omega_1^j & \cdots & \omega_k^j & \cdots & \omega_{n-1}^j \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \cdots & \omega_k^{n-1} & \cdots & \omega_{n-1}^{n-1} \end{vmatrix} = \mathbf{M};$$

- For $\ell = 1, \dots, n-1$ and $\ell \neq j$, we have:

$$\sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^\ell = \begin{vmatrix} \omega_1^1 & \cdots & \omega_k^1 & \cdots & \omega_{n-1}^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \omega_1^{j-1} & \cdots & \omega_k^{j-1} & \cdots & \omega_{n-1}^{j-1} \\ \omega_1^\ell & \cdots & \omega_k^\ell & \cdots & \omega_{n-1}^\ell \\ \omega_1^{j+1} & \cdots & \omega_k^{j+1} & \cdots & \omega_{n-1}^{j+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \cdots & \omega_k^{n-1} & \cdots & \omega_{n-1}^{n-1} \end{vmatrix} = 0,$$

because this is the determinant of a matrix with two equal rows.

- For $\ell = n, \dots, m$, we have:

$$\begin{aligned}
\text{cof}(\omega_n^j, \mathbf{M}_\ell) &= (-1)^{n+j} \begin{vmatrix} \omega_1^1 & \cdots & \omega_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \omega_1^{j-1} & \cdots & \omega_{n-1}^{j-1} \\ \omega_1^{j+1} & \cdots & \omega_{n-1}^{j+1} \\ \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \cdots & \omega_{n-1}^{n-1} \\ \omega_1^\ell & \cdots & \omega_{n-1}^\ell \end{vmatrix} \\
&= (-1)^{n+j} \sum_{k=1}^{n-1} \omega_k^\ell (-1)^{n-1+k} \begin{vmatrix} \omega_1^1 & \cdots & \omega_{k-1}^1 & \omega_{k+1}^1 & \cdots & \omega_{n-1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_1^{j-1} & \cdots & \omega_{k-1}^{j-1} & \omega_{k+1}^{j-1} & \cdots & \omega_{n-1}^{j-1} \\ \omega_1^{j+1} & \cdots & \omega_{k-1}^{j+1} & \omega_{k+1}^{j+1} & \cdots & \omega_{n-1}^{j+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \cdots & \omega_{k-1}^{n-1} & \omega_{k+1}^{n-1} & \cdots & \omega_{n-1}^{n-1} \end{vmatrix} \\
&= \sum_{k=1}^{n-1} (-1)^{j-1+k} \omega_k^\ell (-1)^{j+k} \text{cof}(\omega_k^j, M) = - \sum_{k=1}^{n-1} \text{cof}(\omega_k^j, M) \omega_k^\ell
\end{aligned}$$

Thus, replacing the rows L'_j , for $j = 1, \dots, n-1$, Matrix (25) becomes

$$(27) \quad \begin{bmatrix} \mathbf{M} & \cdots & 0 & \vdots & -\text{cof}(\omega_n^1, \mathbf{M}_n) & \cdots & -\text{cof}(\omega_n^1, \mathbf{M}_m) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{M} & \vdots & -\text{cof}(\omega_n^{n-1}, \mathbf{M}_n) & \cdots & -\text{cof}(\omega_n^{n-1}, \mathbf{M}_m) \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_{n-1}} & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_p} & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{M}_m}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_{n-1}} & \vdots & \frac{\partial \mathbf{M}_m}{\partial x_p} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m} \end{bmatrix}.$$

that still has maximal rank. Let us denote the first $n-1$ row vectors of Matrix (27) by L''_j , for $j = 1, \dots, n-1$, and let us consider the following expression for $j = n, \dots, m$,

$$\begin{aligned}
&\mathbf{M} L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k \\
&= \mathbf{M} \left(\frac{\partial \mathbf{M}_j}{\partial x_1}, \dots, \frac{\partial \mathbf{M}_j}{\partial x_{n-1}}, \frac{\partial \mathbf{M}_j}{\partial x_n}, \dots, \frac{\partial \mathbf{M}_j}{\partial x_m} \right) \\
&+ \left(-\mathbf{M} \frac{\partial \mathbf{M}_j}{\partial x_1}, \dots, -\mathbf{M} \frac{\partial \mathbf{M}_j}{\partial x_{n-1}}, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_n), \dots, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_m) \right) \\
&= \left(0, \dots, 0, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_n) + \mathbf{M} \frac{\partial \mathbf{M}_j}{\partial x_n}, \dots, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_m) + \mathbf{M} \frac{\partial \mathbf{M}_j}{\partial x_m} \right).
\end{aligned}$$

Note that $\mathbf{M} = \text{cof}(\omega_n^i, \mathbf{M}_i)$, for $i = n, \dots, m$, so that the expression

$$\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k$$

is equal to

$$\left(0, \dots, 0, \sum_{k \in I_n} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_n), \dots, \sum_{k \in I_m} \frac{\partial \mathbf{M}_j}{\partial x_k} \text{cof}(\omega_n^k, M_m) \right).$$

Thus, by Equation (24), we obtain

$$\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k = (0, \dots, 0, m_{nj}, \dots, m_{mj}).$$

For $j = n, \dots, m$, we replace the row L'_j in Matrix (27) by

$$\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k,$$

such that the matrix obtained:

$$(28) \quad \begin{bmatrix} \mathbf{M} & \cdots & 0 & \vdots & -\text{cof}(\omega_n^1, \mathbf{M}_n) & \cdots & -\text{cof}(\omega_n^1, \mathbf{M}_m) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{M} & \vdots & -\text{cof}(\omega_n^{n-1}, \mathbf{M}_n) & \cdots & -\text{cof}(\omega_n^{n-1}, \mathbf{M}_m) \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & & \vdots & & & \\ & O_{(n-1)} & & \vdots & & & (-\mathbf{M}')^t \\ & & & \vdots & & & \end{bmatrix}$$

also is non-singular. So, since $\mathbf{M} \neq 0$, we have that $\det \mathbf{M}' \neq 0$.

Thus, we can conclude that Matrix (22) is non-singular if and only if Matrix (23) is non-singular, which occurs if and only if

$$\det \begin{bmatrix} d_x \xi_1(p) \\ \vdots \\ d_x \xi_m(p) \end{bmatrix} \neq 0.$$

Therefore, p will be a non-degenerate zero of $\xi|_{\Sigma^1(\omega)}$ if and only if p is a non-degenerate zero of ξ . \square

Lemma 4.10. *Let $p \in A_{k+1}(\omega)$ be a zero of $\xi|_{\Sigma^{k+1}(\omega)}$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, p is a non-degenerate zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if p is a non-degenerate zero of $\xi|_{\Sigma^k(\omega)}$.*

Proof. Let $p \in A_{k+1}(\omega)$ be a zero of $\xi|_{\Sigma^{k+1}(\omega)}$ and let $\mathcal{U} \subset M$ be an open neighborhood of p at which $\mathbf{M}(x) \neq 0$ and the singular sets $\Sigma^k(\omega)$ ($k = 2, \dots, n$) are defined by $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}$. By Szafraniec's characterization ([16, p. 196]), p is a zero of the restriction $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if there exists a unique $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_{k+1}) \in \mathbb{R}^{m-n+k+1}$ such that

$$\xi(p) + \sum_{j=n}^m \lambda_j d\mathbf{M}_j(p) + \sum_{j=2}^{k+1} \beta_j d\Delta_j(p) = 0.$$

Since p is a zero of $\xi|_{\Sigma^k(\omega)}$, we have $\beta_{k+1} = 0$. Moreover, also by Szafraniec's characterization, p is a non-degenerate zero of $\xi|_{\Sigma^{k+1}(\omega)}$ if and only if the determinant of the following matrix does not vanish at p :

$$(29) \quad \begin{bmatrix} \text{Jac}_x \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i + \sum_{j=2}^k \beta_j d\Delta_j \right) & \frac{\partial \mathbf{M}_n}{\partial x_1} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \dots & \frac{\partial \Delta_k}{\partial x_1} & \frac{\partial \Delta_{k+1}}{\partial x_1} \\ & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & \frac{\partial \mathbf{M}_n}{\partial x_m} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \dots & \frac{\partial \Delta_k}{\partial x_m} & \frac{\partial \Delta_{k+1}}{\partial x_m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & d_x \mathbf{M}_n & & & & & & \\ & \vdots & & & & & & \\ & d_x \mathbf{M}_m & & & & & & \\ & d_x \Delta_2 & & & & & O_{(m-n+k+1)} & \\ & \vdots & & & & & & \\ & d_x \Delta_{k+1} & & & & & & \end{bmatrix}.$$

Analogously, p is a non-degenerate zero of $\xi|_{\Sigma^k(\omega)}$ if and only if the determinant of the following matrix does not vanish at p :

$$(30) \quad \begin{bmatrix} & \frac{\partial \mathbf{M}_n}{\partial x_1} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \dots & \frac{\partial \Delta_k}{\partial x_1} \\ \text{Jac}_x \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i + \sum_{j=2}^k \beta_j d\Delta_j \right) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ & \frac{\partial \mathbf{M}_n}{\partial x_m} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \dots & \frac{\partial \Delta_k}{\partial x_m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & d_x \mathbf{M}_n & & & & & \\ & \vdots & & & & & \\ & d_x \mathbf{M}_m & & & & & \\ & d_x \Delta_2 & & & & & O_{(m-n+k)} \\ & \vdots & & & & & \\ & d_x \Delta_k & & & & & \end{bmatrix}.$$

Thus, to prove the lemma it is enough to show that the Matrix (29) is non-singular at p if and only if the Matrix (30) is non-singular at p .

Note that the Jacobian matrix with respect to x

$$(31) \quad \text{Jac}_x \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i + \sum_{j=2}^k \beta_j d\Delta_j \right)$$

is a submatrix of both the Matrices (29) and (30). And remember that, for x in an open neighborhood of p , $\Delta_{k+1} = \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k, \Omega_1, \dots, \Omega_{n-k})$, where $\{\Omega_1(x), \dots, \Omega_{n-k}(x)\}$ is a basis of a vector subspace supplementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ in $\langle \bar{\omega}(x) \rangle$. Thus,

$$\langle \bar{\omega}(x) \rangle = \langle \Omega_1(x), \dots, \Omega_{n-k}(x) \rangle \oplus (\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)).$$

Since, for almost every a , $\xi|_{\Sigma^{k-1}(\omega)}(p) \neq 0$ then $\xi(p) \in \langle \bar{\omega}(p) \rangle \setminus N_p^* \Sigma^{k-1}(\omega)$. That is, there exists $(\mu_1, \dots, \mu_{n-k}) \in \mathbb{R}^{n-k} \setminus \{\vec{0}\}$ so that

$$\xi(p) = \sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \varphi(p),$$

for some $\varphi(p) \in N_p^* \Sigma^{k-1}(\omega)$, where $\varphi(p) = \sum_{i=n}^m \tilde{\lambda}_i d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j(p)$. Thus,

$$(32) \quad \xi(p) = \sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \sum_{i=n}^m \tilde{\lambda}_i d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j(p),$$

and the expression

$$\xi(p) + \sum_{j=n}^m \lambda_j d\mathbf{M}_j(p) + \sum_{j=2}^k \beta_j d\Delta_j(p)$$

can be written as:

$$(33) \quad \sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(p) + \beta_k d\Delta_k(p).$$

Let us consider the mapping

$$H(x) = \sum_{i=1}^{n-k} \mu_i \Omega_i(x) + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(x) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(x) + \beta_k d\Delta_k(x),$$

defined on an open neighborhood $\mathcal{U} \subset M$ of p , which is equal to

$$\sum_{i=1}^{n-k} \mu_i \begin{bmatrix} \Omega_i^1 \\ \vdots \\ \Omega_i^m \end{bmatrix} + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) \begin{bmatrix} \frac{\partial \mathbf{M}_i}{\partial x_1} \\ \vdots \\ \frac{\partial \mathbf{M}_i}{\partial x_m} \end{bmatrix} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) \begin{bmatrix} \frac{\partial \Delta_j}{\partial x_1} \\ \vdots \\ \frac{\partial \Delta_j}{\partial x_m} \end{bmatrix} + \beta_k \begin{bmatrix} \frac{\partial \Delta_k}{\partial x_1} \\ \vdots \\ \frac{\partial \Delta_k}{\partial x_m} \end{bmatrix}$$

and can be written as

$$\begin{bmatrix} \sum_{i=1}^{n-k} \mu_i \Omega_i^1 + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) \frac{\partial \mathbf{M}_i}{\partial x_1} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) \frac{\partial \Delta_j}{\partial x_1} + \beta_k \frac{\partial \Delta_k}{\partial x_1} \\ \vdots \\ \sum_{i=1}^{n-k} \mu_i \Omega_i^m + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) \frac{\partial \mathbf{M}_i}{\partial x_m} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) \frac{\partial \Delta_j}{\partial x_m} + \beta_k \frac{\partial \Delta_k}{\partial x_m} \end{bmatrix}.$$

Then, the Jacobian matrix of $H(x)$ is given by:

$$(34) \quad \begin{bmatrix} \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^1 + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) d_x \frac{\partial \mathbf{M}_i}{\partial x_1} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \frac{\partial \Delta_j}{\partial x_1} + \beta_k d_x \frac{\partial \Delta_k}{\partial x_1} \\ \vdots \\ \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^m + \sum_{i=n}^m (\lambda_i + \tilde{\lambda}_i) d_x \frac{\partial \mathbf{M}_i}{\partial x_m} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \frac{\partial \Delta_j}{\partial x_m} + \beta_k d_x \frac{\partial \Delta_k}{\partial x_m} \end{bmatrix}.$$

To apply the Lemma 4.1, fix the notation: $A_i(x) = (a_{1i}(x), \dots, a_{mi}(x))$, such that

$$A_i(x) := \begin{cases} \Omega_i(x), & i = 1, \dots, n-k; \\ d\mathbf{M}_i(x), & i = n, \dots, m; \end{cases}$$

$$A_{n-k+j-1}(x) := d\Delta_j(x), \quad j = 2, \dots, k;$$

$$\alpha_i := \begin{cases} \mu_i, & i = 1, \dots, n-k; \text{ (we can suppose } \alpha_1 \neq 0, \text{ since } \xi(p) \neq \varphi(p)) \\ (\lambda_i + \tilde{\lambda}_i), & i = n, \dots, m; \end{cases}$$

$$\alpha_{n-k+j-1} := (\beta_j + \tilde{\beta}_j), \quad j = 2, \dots, k; \quad (\tilde{\beta}_k = 0).$$

Since $\xi(p) + \sum_{j=n}^m \lambda_j d\mathbf{M}_j(p) + \sum_{j=2}^{k+1} \beta_j d\Delta_j(p) = 0$, by Expression (33) we have

$$\sum_{i=1}^m \alpha_i A_i(p) = 0.$$

That is,

$$\sum_{i=1}^m \alpha_i a_{ji}(p) = 0, \quad \forall j = 1, \dots, m.$$

Then, applying Lemma 4.1 we know that

$$(35) \quad \alpha_1 \operatorname{cof}(a_{ik}(p)) - \alpha_k \operatorname{cof}(a_{i1}(p)) = 0, \quad \forall i, k = 1, \dots, m.$$

We also have that

$$\begin{aligned} \Delta_{k+1} &= \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k, \Omega_1, \dots, \Omega_{n-k}) \\ &= \det(A_n, \dots, A_m, A_{n-k+1}, \dots, A_{n-1}, A_1, \dots, A_{n-k}) \\ &= (-1)^\varepsilon \det(A_1, \dots, A_m) \end{aligned}$$

where ε is either equal to zero or equal to 1, depending on the number of required permutations between the column vectors of the matrix

$$(A_n, \dots, A_m, A_{n-k+1}, \dots, A_{n-1}, A_1, \dots, A_{n-k})$$

in order to obtain the matrix (A_1, \dots, A_m) . Thus,

$$\begin{aligned} (-1)^\varepsilon d\Delta_{k+1} &= \sum_{i,j=1}^m \operatorname{cof}(a_{ij}) da_{ij} \\ &= \sum_{i=1}^m \left(\operatorname{cof}(a_{i1}) da_{i1} + \sum_{j=2}^m \operatorname{cof}(a_{ij}) da_{ij} \right) \\ &\stackrel{\alpha_1 \neq 0}{=} \sum_{i=1}^m \left(\operatorname{cof}(a_{i1}) da_{i1} + \sum_{j=2}^m \frac{\alpha_j}{\alpha_1} \operatorname{cof}(a_{i1}) da_{ij} \right) \end{aligned}$$

which implies that, for each $x \in \mathcal{U}$

$$\begin{aligned} (-1)^\varepsilon \alpha_1 d\Delta_{k+1} &= \sum_{i=1}^m \left(\alpha_1 \operatorname{cof}(a_{i1}) da_{i1} + \sum_{j=2}^m \alpha_j \operatorname{cof}(a_{i1}) da_{ij} \right) \\ (36) \quad &= \sum_{i=1}^m \operatorname{cof}(a_{i1}) \left[\sum_{j=1}^m \alpha_j da_{ij} \right] \\ &= \sum_{i=1}^m \operatorname{cof}(a_{i1}) \mathcal{L}_i \end{aligned}$$

where $\mathcal{L}_i, i = 1, \dots, m$, denote the rows of the Jacobian matrix (34). If we denote by $\tilde{L}_i, i = 1, \dots, m$, the row vectors of Jacobian matrix (31) at p , then we can verify that

$$(37) \quad \sum_{i=1}^m \text{cof}(a_{i1}) \mathcal{L}_i = \sum_{i=1}^m \text{cof}(a_{i1}) \tilde{L}_i$$

Let us denote the first m row vectors of Matrix (29) by $L_i, i = 1, \dots, m$, and its last row vector by $L_{\Delta_{k+1}}$. Based on Expressions (36) at p and (37), we replace the row vector $L_{\Delta_{k+1}}$ by

$$(38) \quad (-1)^\varepsilon \alpha_1 L_{\Delta_{k+1}} - \sum_{i=1}^m \text{cof}(a_{i1}) L_i,$$

in order to obtain a new last row vector given by

$$L_{\Delta_{k+1}} := (\underbrace{0, \dots, 0}_m, \gamma_n, \dots, \gamma_m, \tilde{\gamma}_2, \dots, \tilde{\gamma}_k, \gamma_{k+1}),$$

where

$$\begin{aligned} \gamma_j &= - \sum_{i=1}^m \text{cof}(a_{i1}) \frac{\partial \mathbf{M}_j}{\partial x_i}, \quad \forall j = n, \dots, m; \\ \tilde{\gamma}_j &= - \sum_{i=1}^m \text{cof}(a_{i1}) \frac{\partial \Delta_j}{\partial x_i}, \quad \forall j = 2, \dots, k+1. \end{aligned}$$

Note that, for $j = n, \dots, m$,

$$\gamma_j = - \sum_{i=1}^m \text{cof}(a_{i1}) \frac{\partial \mathbf{M}_j}{\partial x_i} = - \begin{vmatrix} \frac{\partial \mathbf{M}_j}{\partial x_1} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{M}_j}{\partial x_m} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

$$\Rightarrow \gamma_j = - \det(A_j, A_2, \dots, A_m) = 0.$$

For $j = 2, \dots, k$,

$$\tilde{\gamma}_j = - \sum_{i=1}^m \text{cof}(a_{i1}) \frac{\partial \Delta_j}{\partial x_i} = - \begin{vmatrix} \frac{\partial \Delta_j}{\partial x_1} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Delta_j}{\partial x_m} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

$$\Rightarrow \tilde{\gamma}_j = - \det(A_{n-k+j-1}, A_2, \dots, A_m) = 0.$$

Therefore, after replacing the vector row $L_{\Delta_{k+1}}$ in the Matrix (29), we obtain

$$(39) \quad \begin{bmatrix} & & & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \dots & \frac{\partial \Delta_k}{\partial x_1} & \vdots & \frac{\partial \Delta_{k+1}}{\partial x_1} \\ \text{Jac} \left(\xi + \sum_{i=n}^m \lambda_i d\mathbf{M}_i + \sum_{j=2}^k \beta_j d\Delta_j \right) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m} & \dots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \dots & \frac{\partial \Delta_k}{\partial x_m} & \vdots & \frac{\partial \Delta_{k+1}}{\partial x_m} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & d_x \mathbf{M}_n & \vdots & & & & & & & \vdots & 0 \\ & & \vdots & \vdots & & & & & & & \vdots & \vdots \\ & & d_x \mathbf{M}_m & \vdots & & & & & & & \vdots & 0 \\ & & d_x \Delta_2 & \vdots & & & O_{(m-n+k-1)} & & & & \vdots & 0 \\ & & \vdots & \vdots & & & & & & & \vdots & \vdots \\ & & d_x \Delta_k & \vdots & & & & & & & \vdots & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \vec{0} & \vdots & & & \vec{0} & & & & \vdots & \gamma_{k+1}^{\sim} \end{bmatrix}.$$

Let us show that $\gamma_{k+1}^{\sim}(p) \neq 0$. We have

$$\begin{aligned} \gamma_{k+1}^{\sim} &= - \sum_{i=1}^m \text{cof}(a_{i1}) \frac{\partial \Delta_{k+1}}{\partial x_i} \\ &= - \det(d\Delta_{k+1}, A_2, \dots, A_m) \\ &= - \det(d\Delta_{k+1}, \Omega_2, \dots, \Omega_{n-k}, d\Delta_2, \dots, d\Delta_k, d\mathbf{M}_n, \dots, d\mathbf{M}_m). \end{aligned}$$

Suppose that $\gamma_{k+1}^{\sim} = 0$. Since each one of the sets $\{\Omega_2(p), \dots, \Omega_{n-k}(p)\}$ and $\{d\Delta_{k+1}(p), d\Delta_2(p), \dots, d\Delta_k(p), d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p)\}$ consist of linearly independent vectors, there exists $j \in \{2, \dots, n-k\}$ such that $\Omega_j(p) \in N_p^* \Sigma^{k+1}(\omega)$. We can suppose that without loss of generality that $j = n-k$, that is,

$$\Omega_{n-k}(p) \in N_p^* \Sigma^{k+1}(\omega) = \langle d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k, d\Delta_{k+1} \rangle.$$

Since $\xi|_{\Sigma^{k+1}}(p) = 0$, we have $\xi(p) \in N_p^* \Sigma^{k+1}(\omega)$. Thus, by Equation (32), we obtain

$$\begin{aligned} \sum_{i=1}^{n-k} \mu_i \Omega_i + \underbrace{\sum_{i=n}^m \tilde{\lambda}_i d\mathbf{M}_i + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j}_{\in N_p^* \Sigma^{k+1}(\omega)} &\in N_p^* \Sigma^{k+1}(\omega) \\ \Rightarrow \sum_{i=1}^{n-k-1} \mu_i \Omega_i = \sum_{i=1}^{n-k} \mu_i \Omega_i - \mu_{n-k} \Omega_{n-k} &\in N_p^* \Sigma^{k+1}(\omega). \end{aligned}$$

Therefore, $\sum_{i=1}^{n-k-1} \mu_i \Omega_i$ and $\mu_{n-k} \Omega_{n-k}$ are linearly independent vectors in the vector subspace $\langle \Omega_1, \dots, \Omega_{n-k} \rangle \cap N_p^* \Sigma^{k+1}(\omega)$. That is,

$$\dim(\langle \Omega_1(p), \dots, \Omega_{n-k}(p) \rangle \cap N_p^* \Sigma^{k+1}(\omega)) \geq 2.$$

Since $\langle \bar{\omega} \rangle = \langle \Omega_1, \dots, \Omega_{n-k} \rangle \oplus (\langle \bar{\omega} \rangle \cap N_p^* \Sigma^{k-1}(\omega))$, we have

$$\begin{aligned} \langle \bar{\omega} \rangle \cap N_p^* \Sigma^{k+1}(\omega) &= \langle \Omega_1, \dots, \Omega_{n-k} \rangle \cap N_p^* \Sigma^{k+1}(\omega) \\ &\oplus \langle \bar{\omega} \rangle \cap N_p^* \Sigma^{k-1}(\omega). \end{aligned}$$

That is,

$$\dim (\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^{k+1}(\omega)) \geq 2 + (k-1) = k+1,$$

which means that $p \in \Sigma^{k+2}(\omega)$. But this is a contradiction, since $p \in A_{k+1}(\omega)$ by hypothesis and $\Sigma^{k+2}(\omega) = \Sigma^{k+1}(\omega) \setminus A_{k+1}(\omega)$. Therefore $\gamma_{k+1}(p) \neq 0$.

Thus, we conclude that the Matrix (29) is non-singular at p if and only if the Matrix (39) is non-singular at p , which occurs if and only if the Matrix (30) is non-singular at the point p . \square

Lemma 4.11. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, if $p \in A_n(\omega)$ then p is a non-degenerate zero of $\xi|_{\Sigma^{n-1}(\omega)}$.*

Proof. We know that if $p \in A_n(\omega)$ then $\xi|_{\Sigma^{n-1}(\omega)}(p) = 0$. By Szafraniec's characterization [17, p.149-151], p is a non-degenerate zero of $\xi|_{\Sigma^{n-1}(\omega)}$ if and only if the following conditions hold:

$$(i) \Delta(p) = \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_{n-1}, \xi)(p) = 0;$$

$$(ii) \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_{n-1}, d\Delta)(p) \neq 0.$$

Condition (i) is clearly satisfied, since $\xi|_{\Sigma^{n-1}(\omega)}(p) = 0$. Let us verify that condition (ii) also holds.

For each $x \in \Sigma^{n-1}(\omega)$ in an open neighborhood $\mathcal{U} \subset M$ of p , let $\{\Omega'(x)\}$ be a smooth basis for a vector subspace supplementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{n-2}(\omega)$ in the vector space $\langle \bar{\omega}(x) \rangle$. Since $\xi(x) \in \langle \bar{\omega}(x) \rangle$, we have

$$\xi(x) = \lambda(x)\Omega'(x) + \varphi(x),$$

where $\lambda(x) \in \mathbb{R}$ and $\varphi(x) \in \langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{n-2}(\omega)$, $\forall x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$.

In particular, if $x \in A_n(\omega)$, we know that, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\xi|_{\Sigma^{n-2}(\omega)}(x) \neq 0$ and, consequently, $\xi(x) \notin N_x^* \Sigma^{n-2}(\omega)$. Thus $\lambda(p) \neq 0$. For all $x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$, we obtain

$$\begin{aligned} \Delta(x) &= \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_{n-1}, \lambda\Omega' + \varphi)(x) \\ &= \lambda(x) \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_{n-1}, \Omega')(x) \\ &= \lambda(x)\Delta_n(x), \end{aligned}$$

with $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Then, by Lemma 2.8 we have

$$\begin{aligned} &\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p) \rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d(\lambda\Delta_n)(p) \rangle. \end{aligned}$$

However, $d(\lambda\Delta_n)(x) = d\lambda(x)\Delta_n(x) + \lambda(x)d\Delta_n(x)$, $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Thus,

$$\begin{aligned} &\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p) \rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta_n(p) \rangle. \end{aligned}$$

Therefore, $\det(d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)) \neq 0$. \square

Lemma 4.12. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the 1-form $\xi|_{\Sigma^k(\omega)}$ admits only non-degenerate zeros, $k \geq 1$.*

Proof. Suppose that $\xi|_{\Sigma^k(\omega)}(p) = 0$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ since $Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega) = \emptyset$ by Lemma 3.6 and $\Sigma^k(\omega) = A_k(\omega) \cup A_{k+1}(\omega) \cup \Sigma^{k+2}(\omega)$.

If $p \in A_k(\omega)$ then $\xi|_{A_k(\omega)}(p) = 0$. Since $\xi|_{A_k(\omega)}$ admits only non-degenerate zeros and $A_k(\omega) \subset \Sigma^k(\omega)$ is a open subset, we conclude that p is a non-degenerate zero of $\xi|_{\Sigma^k(\omega)}$.

If $p \in A_{k+1}(\omega)$ and $k < n - 1$ then $\xi|_{\Sigma^{k+1}(\omega)}(p) = 0$. In particular, since $A_{k+1}(\omega) \subset \Sigma^{k+1}(\omega)$ is an open subset then $\xi|_{A_{k+1}(\omega)}(p) = 0$. By Lemmas 4.8 and 4.7, $\xi|_{A_{k+1}(\omega)}$ admits only non-degenerate zeros, and since $A_{k+1}(\omega)$ is an open set of $\Sigma^{k+1}(\omega)$, we conclude that p is a non-degenerate zero of $\xi|_{\Sigma^{k+1}(\omega)}$. Therefore, by Lemma 4.10, p is non-degenerate zero of $\xi|_{\Sigma^k(\omega)}$. Finally, if $p \in A_n(\omega)$, by Lemma 4.11, p is a non-degenerate zero of $\xi|_{\Sigma^{n-1}(\omega)}$. \square

Theorem 4.1. *Let $\omega = (\omega_1, \dots, \omega_n)$ be a Morin n -coframe defined on an m -dimensional compact manifold M . Then,*

$$\chi(M) \equiv \sum_{k=1}^n \chi(\overline{A_k(\omega)}) \pmod{2}.$$

Proof. Let us denote by $Z(\varphi)$ the set of zeros of a 1-form φ and let us denote by $\#Z(\varphi)$ the number of elements of this set, whenever $Z(\varphi)$ is finite. Let

$$\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$$

be a 1-form with $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ satisfying the generic conditions of the previous lemmas of Sections 3 and 4.

Since M is compact and the submanifolds $\Sigma^k(\omega)$ are closed in M , by the Poincaré-Hopf Theorem for 1-forms we obtain

- $\chi(M) \equiv \#Z(\xi) \pmod{2}$;
- $\chi(\overline{A_k(\omega)}) = \chi(\Sigma^k(\omega)) \equiv \#Z(\xi|_{\Sigma^k(\omega)}) \pmod{2}$, for $k = 1, \dots, n-1$;
- $\chi(\overline{A_n(\omega)}) = \chi(\Sigma^n(\omega)) \equiv \#Z(\xi|_{\Sigma^n(\omega)}) \pmod{2}$.

By Lemma 3.1, if $p \in Z(\xi)$ then $p \in \Sigma^1(\omega)$ and $\xi|_{\Sigma^1(\omega)}(p) = 0$. Moreover, by Lemma 3.5, $Z(\xi) \cap \Sigma^2(\omega) = \emptyset$. Thus $p \in A_1(\omega)$. On the other hand, Lemma 3.2 shows that if $p \in Z(\xi|_{\Sigma^1(\omega)}) \cap A_1(\omega)$ then p is also a zero of the 1-form ξ . Thus,

$$\#Z(\xi) \equiv \#Z(\xi|_{\Sigma^1(\omega)} \cap A_1(\omega)) \pmod{2}.$$

By Lemma 3.6, if $p \in Z(\xi|_{\Sigma^k(\omega)})$ then $p \notin \Sigma^{k+2}(\omega)$. Thus, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ and, for $k = 1, \dots, n-1$, we have

$$\#Z(\xi|_{\Sigma^k(\omega)}) \equiv \#Z(\xi|_{\Sigma^k(\omega)} \cap A_k(\omega)) + \#Z(\xi|_{\Sigma^k(\omega)} \cap A_{k+1}(\omega)) \pmod{2}.$$

By Lemma 3.2, we also have

$$\#Z(\xi|_{\Sigma^k(\omega)} \cap A_{k+1}(\omega)) = \#Z(\xi|_{\Sigma^{k+1}(\omega)} \cap A_{k+1}(\omega))$$

and by Lemma 3.3,

$$\#A_n(\omega) = \#Z(\xi|_{\Sigma^{n-1}(\omega)} \cap A_n(\omega)).$$

Then,

- $\chi(M) \equiv \#Z(\xi|_{\Sigma^1(\omega)} \cap A_1(\omega)) \pmod{2}$;
- For $k = 1, \dots, n-1$,
 $\chi(\overline{A_k(\omega)}) \equiv \#Z(\xi|_{\Sigma^k(\omega)} \cap A_k(\omega)) + \#Z(\xi|_{\Sigma^{k+1}(\omega)} \cap A_{k+1}(\omega)) \pmod{2}$;
- $\chi(\overline{A_n(\omega)}) = \#Z(\xi|_{\Sigma^{n-1}(\omega)} \cap A_n(\omega)).$

Therefore,

$$\begin{aligned}
\chi(M) + \sum_{k=1}^n \chi(\overline{A_k(\omega)}) &\equiv 2\#Z(\xi_{|\Sigma^1(\omega)} \cap A_1(\omega)) \\
&+ 2\#Z(\xi_{|\Sigma^2(\omega)} \cap A_2(\omega)) + \dots \\
&+ 2\#Z(\xi_{|\Sigma^{n-1}(\omega)} \cap A_{n-1}(\omega)) \\
&+ 2\#Z(\xi_{|\Sigma^n(\omega)} \cap A_n(\omega)) \pmod{2} \\
&\equiv 0 \pmod{2}.
\end{aligned}$$

□

As for the definition of Morin n -coframes, the results presented in Sections 3 and 4 of this paper also can be naturally adapted to the context of n -frames. In particular, the main theorems that have been used, as the Poincaré-Hopf Theorem and the Szafranec's characterization, have their respective versions for vector fields.

Finally, we end the paper with a very simple example. Let us verify that Theorem 4.1 indeed holds for the Morin 2-frame $V = (V_1, V_2)$ presented in the Example 2.5. To do this, it is enough to see that the torus T is a compact manifold with $\chi(T) = 0$. Moreover, $\overline{A_1(V)} = \Sigma^1(V)$ is given by two circles in \mathbb{R}^3 and $\overline{A_2(V)}$ consists of four points, so that $\chi(\overline{A_1(V)}) = 0$ and $\chi(\overline{A_2(V)}) = 4$. Therefore,

$$\chi(T) \equiv \chi(\overline{A_1(V)}) + \chi(\overline{A_2(V)}) \pmod{2}.$$

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